# CS 473 ✧ Spring 2017

# Midterm 2

**April 4, 2016**

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1. Let $G = (V, E)$ be an arbitrary undirected graph. Suppose we color each vertex of $G$ uniformly and independently at random from a set of three colors: red, green, or blue. An edge of $G$ is *monochromatic* if both of its endpoints have the same color.

   (a) What is the *exact* expected number of monochromatic edges? (Your answer should be a simple function of $V$ and $E$.)

   (b) For each edge $e \in E$, define an indicator variable $X_e$ that equals 1 if $e$ is monochromatic and 0 otherwise. ***Prove*** that
   $$\Pr[(X_a = 1) \wedge (X_b = 1)] \;=\; \Pr[X_a = 1] \cdot \Pr[X_b = 1]$$
   for every pair of edges $a \neq b$. This claim implies that the random variables $X_e$ are pairwise independent.

   (c) ***Prove*** that there is a graph $G$ such that
   $$\Pr[(X_a = 1) \wedge (X_b = 1) \wedge (X_c = 1)] \;\neq\; \Pr[X_a = 1] \cdot \Pr[X_b = 1] \cdot \Pr[X_c = 1]$$
   for some triple of distinct edges $a, b, c$ in $G$. This claim implies that the random variables $X_e$ are *not necessarily* 3-wise independent.

2. The White Rabbit has a very poor memory, and so he is constantly forgetting his regularly scheduled appointments with the Queen of Hearts. In an effort to avoid further beheadings of court officials, The King of Hearts has installed an app on Rabbit's pocket watch to automatically remind Rabbit of any upcoming appointments. For each reminder Rabbit receives, Rabbit has a 50% chance of actually remembering his appointment (decided by an independent fair coin flip).

   First, suppose the King of Hearts sends Rabbit $k$ separate reminders for a *single* appointment.

   (a) What is the *exact* probability that Rabbit will remember his appointment? Your answer should be a simple function of $k$.

   (b) What value of $k$ should the King choose so that the probability that Rabbit will remember this appointment is at least $1 - 1/n^\alpha$? Your answer should be a simple function of $n$ and $\alpha$.

   Now suppose the King of Hearts sends Rabbit $k$ separate reminders for each of $n$ different appointments. (That's $nk$ reminders altogether.)

   (c) What is the *exact* expected number of appointments that Rabbit will remember? Your answer should be a simple function of $n$ and $k$.

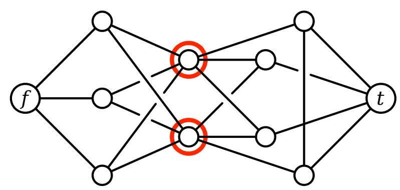
   (d) What value of $k$ should the King choose so that the probability that Rabbit remembers *every* appointment is at least $1 - 1/n^\alpha$? Again, your answer should be a simple function of $n$ and $\alpha$.

   *[Hint: There is a simple solution that does not use tail inequalities.]*

3. The Island of Sodor is home to an extensive rail network. Recently, several cases of a deadly contagious disease (either swine flu or zombies; reports are unclear) have been reported in the village of Ffarquhar. The controller of the Sodor railway plans to close certain railway stations to prevent the disease from spreading to Tidmouth, his home town. No trains can pass through a closed station. To minimize expense (and public notice), he wants to close as few stations as possible. However, he cannot close the Ffarquhar station, because that would expose him to the disease, and he cannot close the Tidmouth station, because then he couldn't visit his favorite pub.

   Describe and analyze an algorithm to find the minimum number of stations that must be closed to block all rail travel from Ffarquhar to Tidmouth. The Sodor rail network is represented by an undirected graph, with a vertex for each station and an edge for each rail connection between two stations. Two special vertices $f$ and $t$ represent the stations in Ffarquhar and Tidmouth.

   For example, given the following input graph, your algorithm should return the integer 2.




4. The Department of Commuter Silence at Shampoo-Banana University has a flexible curriculum with a complex set of graduation requirements. The department offers $n$ different courses, and there are $m$ different requirements. Each requirement specifies a subset of the $n$ courses and the number of courses that must be taken from that subset. The subsets for different requirements may overlap, but each course can only be used to satisfy *at most one* requirement.

   For example, suppose there are $n = 5$ courses $A, B, C, D, E$ and $m = 2$ graduation requirements:

   - You must take at least 2 courses from the subset $\{A, B, C\}$.
   - You must take at least 2 courses from the subset $\{C, D, E\}$.

   Then a student who has only taken courses $B, C, D$ cannot graduate, but a student who has taken either $A, B, C, D$ or $B, C, D, E$ can graduate.

   Describe and analyze an algorithm to determine whether a given student can graduate. The input to your algorithm is the list of $m$ requirements (each specifying a subset of the $n$ courses and the number of courses that must be taken from that subset) and the list of courses the student has taken.