CS 473 ✧ Spring 2016

Homework 4

Due Tuesday, March 1, 2016, at 8pm

Unless a problem specifically states otherwise, you may assume a function RANDOM that takes a positive integer k as input and returns an integer chosen uniformly and independently at random from $\{1, 2, \ldots, k\}$ in $O(1)$ time. For example, to flip a fair coin, you could call RANDOM(2).

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1. Suppose we are given a two-dimensional array $M[1..n, 1..n]$ in which every row and every column is sorted in increasing order and no two elements are equal.

   (a) Describe and analyze an algorithm to solve the following problem in $O(n)$ time: Given indices i, j, i', j' as input, compute the number of elements of M larger than $M[i, j]$ and smaller than $M[i', j']$.

   (b) Describe and analyze an algorithm to solve the following problem in $O(n)$ time: Given indices i, j, i', j' as input, return an element of M chosen uniformly at random from the elements larger than $M[i, j]$ and smaller than $M[i', j']$. Assume the requested range is always non-empty.

   (c) Describe and analyze a randomized algorithm to compute the median element of M in $O(n \log n)$ expected time.

2. ***Tabulated hashing*** uses tables of random numbers to compute hash values. Suppose $|\mathcal{U}| = 2^w \times 2^w$ and $m = 2^\ell$, so the items being hashed are pairs of w-bit strings (or $2w$-bit strings broken in half) and hash values are ℓ-bit strings.

   Let $A[0..2^w - 1]$ and $B[0..2^w - 1]$ be arrays of independent random ℓ-bit strings, and define the hash function $h_{A,B} : \mathcal{U} \to [m]$ by setting

   $$h_{A,B}(x, y) := A[x] \oplus B[y]$$

   where $\oplus$ denotes bit-wise exclusive-or. Let $\mathcal{H}$ denote the set of all possible functions $h_{A,B}$. Filling the arrays A and B with independent random bits is equivalent to choosing a hash function $h_{A,B} \in \mathcal{H}$ uniformly at random.

   (a) Prove that $\mathcal{H}$ is 2-uniform.

   (b) Prove that $\mathcal{H}$ is 3-uniform. *[Hint: Solve part (a) first.]*

   (c) Prove that $\mathcal{H}$ is ***not*** 4-uniform.

   Yes, "see part (b)" is worth full credit for part (a), but only if your solution to part (b) is correct.