CS 473 ✧ Spring 2016

Homework 2

Due Tuesday, February 9, 2016, at 8pm

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1. *[Insert amusing story about distributing polling stations or cell towers or Starbucks or something on a long straight road in rural Iowa. Ha ha ha, how droll.]*

   More formally, you are given a sorted array $X[1..n]$ of distinct numbers and a positive integer k. A set of k intervals ***covers*** X if every element of X lies inside one of the k intervals. Your aim is to find k intervals $[a_1, z_1], [a_2, z_2], \ldots, [a_k, z_k]$ that cover X where the function $\sum_{i=1}^{k}(z_i - a_i)^2$ is as small as possible. Intuitively, you are trying to cover the points with k intervals whose lengths are as close to equal as possible.

   (a) Describe an algorithm that finds k intervals with minimum total squared length that cover X. The running time of your algorithm should be a simple function of n and k.

   (b) Consider the two-dimensional matrix $M[1..n, 1..n]$ defined as follows:

   $$M[i,j] = \begin{cases} (X[j] - X[i])^2 & \text{if } i \le j \\ \infty & \text{otherwise} \end{cases}$$

   Prove that M satisfies the ***Monge property***: $M[i,j] + M[i',j'] \le M[i,j'] + M[i',j]$ for all indices $i < i'$ and $j < j'$.

   (c) ***[Extra credit]*** Describe an algorithm that finds k intervals with minimum total squared length that cover X ***in $O(nk)$ time***. *[Hint: Solve part (a) first, then use part (b).]*

   We strongly recommend submitting your solution to part (a) separately, and only describing your changes to that solution for part (c).

2. The Doctor and River Song decide to play a game on a directed acyclic graph G, which has one source s and one sink t.[1]

   Each player has a token on one of the vertices of G. At the start of the game, The Doctor's token is on the source vertex s, and River's token is on the sink vertex t. The players alternate turns, with The Doctor moving first. On each of his turns, the Doctor moves his token forward along a directed edge; on each of her turns, River moves her token *backward* along a directed edge.

   If the two tokens ever meet on the same vertex, River wins the game. ("Hello, Sweetie!") If the Doctor's token reaches t or River's token reaches s before the two tokens meet, then the Doctor wins the game.

   Describe and analyze an algorithm to determine who wins this game, assuming both players play perfectly. That is, if the Doctor can win *no matter how River moves*, then your algorithm should output "Doctor", and if River can win *no matter how the Doctor moves*, your algorithm should output "River". (Why are these the only two possibilities?) The input to your algorithm is the graph G.

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[1] possibly short for the Untempered **S**chism and the **T**ime Vortex, or the Shining World of the Seven Systems (otherwise known as Gallifrey) and Trenzalore, or Skaro and Telos, or something timey-wimey.