Write your answers in the separate answer booklet.

1. **Multiple Choice:** Each question below has one of the following answers.

   A: $\Theta(1)$ B: $\Theta(\log n)$ C: $\Theta(n)$ D: $\Theta(n \log n)$ E: $\Theta(n^2)$ X: I don't know.

   For each question, write the letter that corresponds to your answer. You do not need to justify your answers. Each correct answer earns you 1 point. Each X earns you $\frac{1}{4}$ point. **Each incorrect answer *costs* you $\frac{1}{2}$ point.** Your total score will be rounded **down** to an integer. Negative scores will be rounded up to zero.

   (a) What is $\sum_{i=1}^{n} \lg i$?

   (b) What is $\sum_{i=1}^{\lg n} i2^i$?

   (c) How many decimal digits are required write the nth Fibonacci number?

   (d) What is the solution of the recurrence $T(n) = 4T(n/8) + n \log n$?

   (e) What is the solution of the recurrence $T(n) = T(n-3) + \frac{5}{n}$?

   (f) What is the solution of the recurrence $T(n) = 5T\left(\left\lceil \frac{n+13}{3} \right\rceil + \lfloor \sqrt{n} \rfloor\right) + (10n-7)^2 - \frac{\lg^3 n}{\lg \lg n}$?

   (g) How long does it take to construct a Huffman code, given an array of n character frequencies as input?

   (h) How long does it take to sort an array of size n using quicksort?

   (i) Given an unsorted array $A[1..n]$, how long does it take to construct a binary search tree for the elements of A?

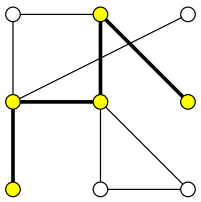
   (j) A train leaves Chicago at 8:00pm and travels south at 75 miles per hour. Another train leaves New Orleans at 1:00pm and travels north at 60 miles per hour. The conductors of both trains are playing a game of chess over the phone. After each player moves, the other player must move before his train has traveled five miles. How many moves do the two players make before their trains pass each other (somewhere near Memphis)?

2. Describe and analyze efficient algorithms to solve the following problems:

   (a) Given a set of n integers, does it contain a pair of elements a, b such that $a + b = 0$?

   (b) Given a set of n integers, does it contain three elements a, b, c such that $a + b = c$?

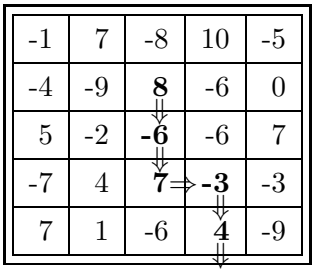
3. A *tonian path* in a graph G is a simple path in G that visits more than half of the vertices of G. (Intuitively, a tonian path is "most of a Hamiltonian path".) Prove that it is NP-hard to determine whether or not a given graph contains a tonian path.



A tonian path in a 9-vertex graph.

4. *Vankin's Mile* is a solitaire game played on an $n \times n$ square grid. The player starts by placing a token on any square of the grid. Then on each turn, the player moves the token either one square to the right or one square down. The game ends when player moves the token off the edge of the board. Each square of the grid has a numerical value, which could be positive, negative, or zero. The player starts with a score of zero; whenever the token lands on a square, the player adds its value to his score. The object of the game is to score as many points as possible.

   For example, given the grid below, the player can score $8 - 6 + 7 - 3 + 4 = 10$ points by placing the initial token on the 8 in the second row, and then moving down, down, right, down, down. (This is *not* the best possible score for these values.)



   Describe and analyze an algorithm to compute the maximum possible score for a game of Vankin's Mile, given the $n \times n$ array of values as input.

5. Suppose you are given two sorted arrays $A[1..m]$ and $B[1..n]$ and an integer k. Describe an algorithm to find the kth smallest element in the union of A and B in $\Theta(\log(m+n))$ time. For example, given the input

$$A[1..8] = [0, 1, 6, 9, 12, 13, 18, 20] \qquad B[1..5] = [2, 5, 8, 17, 19] \qquad k = 6$$

   your algorithm should return 8. You can assume that the arrays contain no duplicates. *[Hint: What can you learn from comparing one element of A to one element of B?]*