CS/ECE 374 ✧ Fall 2016

Homework 9

Due Tuesday, November 15, 2016 at 8pm

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1. Consider the following problem, called BoxDepth: Given a set of n axis-aligned rectangles in the plane, how big is the largest subset of these rectangles that contain a common point?

   (a) Describe a polynomial-time reduction from BoxDepth to MaxClique, and prove that your reduction is correct.

   (b) Describe and analyze a polynomial-time algorithm for BoxDepth. *[Hint: Don't try to optimize the running time; $O(n^3)$ is good enough.]*

   (c) Why don't these two results imply that P=NP?

2. This problem asks you to describe polynomial-time reductions between two closely related problems:

   - SubsetSum: Given a set S of positive integers and a target integer T, is there a subset of S whose sum is T?
   - Partition: Given a set S of positive integers, is there a way to partition S into two subsets S_1 and S_2 that have the same sum?

   (a) Describe a polynomial-time reduction from SubsetSum to Partition.

   (b) Describe a polynomial-time reduction from Partition to SubsetSum.

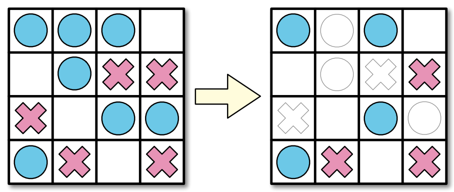
   Don't forget to to prove that your reductions are correct.

3. Suppose you are given a graph $G = (V, E)$ where V represents a collection of people and an edge between two people indicates that they are friends. You wish to partition V into at most k non-overlapping groups $V_1, V_2, \ldots, V_k$ such that each group is very cohesive. One way to model cohesiveness is to insist that each pair of people in the same group should be friends; in other words, they should form a clique.

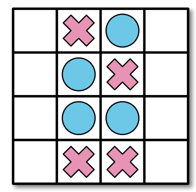
   Prove that the following problem is NP-hard: Given an undirected graph G and an integer k, decide whether the vertices of G can be partitioned into k cliques.

Solved Problem

4. Consider the following solitaire game. The puzzle consists of an $n \times m$ grid of squares, where each square may be empty, occupied by a red stone, or occupied by a blue stone. The goal of the puzzle is to remove some of the given stones so that the remaining stones satisfy two conditions: (1) every row contains at least one stone, and (2) no column contains stones of both colors. For some initial configurations of stones, reaching this goal is impossible.



A solvable puzzle and one of its many solutions.



An unsolvable puzzle.

Prove that it is NP-hard to determine, given an initial configuration of red and blue stones, whether this puzzle can be solved.

Solution: We show that this puzzle is NP-hard by describing a reduction from 3SAT.

Let Φ be a 3CNF boolean formula with m variables and n clauses. We transform this formula into a puzzle configuration in polynomial time as follows. The size of the board is $n \times m$. The stones are placed as follows, for all indices i and j:

- If the variable x_j appears in the ith clause of Φ, we place a blue stone at (i, j).
- If the negated variable $\overline{x_j}$ appears in the ith clause of Φ, we place a red stone at (i, j).
- Otherwise, we leave cell (i, j) blank.

We claim that this puzzle has a solution if and only if Φ is satisfiable. This claim immediately implies that solving the puzzle is NP-hard. We prove our claim as follows:

$\Longrightarrow$ First, suppose Φ is satisfiable; consider an arbitrary satisfying assignment. For each index j, remove stones from column j according to the value assigned to x_j:

- If $x_j = \text{True}$, remove all red stones from column j.
- If $x_j = \text{False}$, remove all blue stones from column j.

In other words, remove precisely the stones that correspond to False literals. Because every variable appears in at least one clause, each column now contains stones of only one color (if any). On the other hand, each clause of Φ must contain at least one True literal, and thus each row still contains at least one stone. We conclude that the puzzle is satisfiable.

$\Longleftarrow$ On the other hand, suppose the puzzle is solvable; consider an arbitrary solution. For each index j, assign a value to x_j depending on the colors of stones left in column j:

- If column j contains blue stones, set $x_j = \text{True}$.
- If column j contains red stones, set $x_j = \text{False}$.
- If column j is empty, set x_j arbitrarily.

In other words, assign values to the variables so that the literals corresponding to the remaining stones are all True. Each row still has at least one stone, so each clause of Φ contains at least one True literal, so this assignment makes $\Phi = \text{True}$. We conclude that Φ is satisfiable.

This reduction clearly requires only polynomial time. ■

Rubric (for all polynomial-time reductions): 10 points =

+ 3 points for the reduction itself
  – For an NP-hardness proof, the reduction must be from a known NP-hard problem. You can use any of the NP-hard problems listed in the lecture notes (except the one you are trying to prove NP-hard, of course).
+ 3 points for the "if" proof of correctness
+ 3 points for the "only if" proof of correctness
+ 1 point for writing "polynomial time"

- An incorrect polynomial-time reduction that still satisfies half of the correctness proof is worth at most 4/10.
- A reduction in the wrong direction is worth 0/10.