"CS 374" Fall 2014 — Homework 0

Due Tuesday, September 2, 2014 at noon

••• Some important course policies •••

- **Each student must submit individual solutions for this homework.** You may use any source at your disposal—paper, electronic, or human—but you ***must*** cite ***every*** source that you use. See the academic integrity policies on the course web site for more details. For all future homeworks, groups of up to three students will be allowed to submit joint solutions.
- **Submit your solutions on standard printer/copier paper.** At the top of each page, please clearly print your name and NetID, and indicate your registered discussion section. Use both sides of the paper. If you plan to write your solutions by hand, please print the last three pages of this homework as templates. If you plan to typeset your homework, you can find a LaTeX template on the course web site; well-typeset homework will get a small amount of extra credit.
- **Submit your solutions in the drop boxes outside 1404 Siebel.** There is a separate drop box for each numbered problem. Don't staple your entire homework together. Don't give your homework to Jeff in class; he is fond of losing important pieces of paper.
- **Avoid the Three Deadly Sins!** There are a few dangerous writing (and thinking) habits that will trigger an automatic zero on any homework or exam problem. Yes, we are completely serious.
  - Give complete solutions, not just examples.
  - Declare all your variables.
  - Never use weak induction.
- Answering any homework or exam problem (or subproblem) in this course with "I don't know" *and nothing else* is worth 25% partial credit. We will accept synonyms like "No idea" or "WTF", but you must write *something*.

See the course web site for more information.

If you have any questions about these policies,
please don't hesitate to ask in class, in office hours, or on Piazza.

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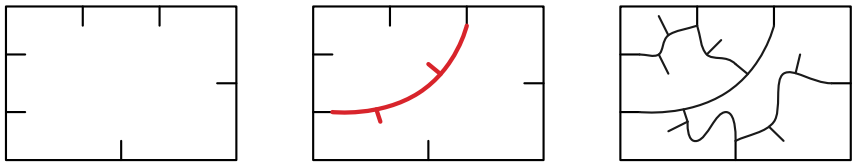
1. The Terminal Game is a two-person game played with pen and paper. The game begins by drawing a rectangle with n “terminals” protruding into the rectangles, for some positive integer n, as shown in the figure below. On a player’s turn, she selects two terminals, draws a simple curve from one to the other without crossing any other curve (or itself), and finally draws a new terminal on each side of the curve. A player loses if it is her turn and no moves are possible, that is, if no two terminals may be connected without crossing at least one other curve.



The initial setup. The first turn. No more moves.

Analyze this game, answering the following questions (and any more that you determine the answers to): When is it better to play first, and when it is better to play second? Is there always a winning strategy? What is the smallest number of moves in which you can defeat your opponent? Prove your answers are correct.

2. Herr Professor Doktor Georg von den Dschungel has a 23-node binary tree, in which each node is labeled with a unique letter of the German alphabet, which is just like the English alphabet with four extra letters: **Ä**, **Ö**, **Ü**, and **ß**. (Don’t confuse these with **A**, **O**, **U**, and **B**!) Preorder and postorder traversals of the tree visit the nodes in the following order:

   - Preorder: **B K Ü E H L Z I Ö R C ß T S O A Ä D F M N U G**
   - Postorder: **H I Ö Z R L E C Ü S O T A ß K D M U G N F Ä B**

   (a) List the nodes in Professor von den Dschungel’s tree in the order visited by an inorder traversal.

   (b) Draw Professor von den Dschungel’s tree.

3. Recursively define a set L of strings over the alphabet $\{\mathbf{0}, \mathbf{1}\}$ as follows:

   - The empty string ε is in L.
   - For any two strings x and y in L, the string $\mathbf{0}x\mathbf{1}y\mathbf{0}$ is also in L.
   - These are the only strings in L.

   (a) Prove that the string **000010101010010100** is in L.

   (b) Prove by induction that every string in L has exactly twice as many **0**s as **1**s.

   (c) Give an example of a string with exactly twice as many **0**s as **1**s that is *not* in L.

   Let $\#(a, w)$ denote the number of times symbol a appears in string w; for example,

   $$\#(\mathbf{0}, \mathbf{000010101010010100}) = 12 \qquad \text{and} \qquad \#(\mathbf{1}, \mathbf{000010101010010100}) = 6.$$

   You may assume without proof that $\#(a, xy) = \#(a, x) + \#(a, y)$ for any symbol a and any strings x and y.

4. **This is an extra credit problem. Submit your solutions in the drop box for problem 2 (but don't staple your solutions for 2 and 4 together).**

   A *perfect riffle shuffle*, also known as a *Faro shuffle*, is performed by cutting a deck of cards exactly in half and then *perfectly* interleaving the two halves. There are two different types of perfect shuffles, depending on whether the top card of the resulting deck comes from the top half or the bottom half of the original deck. An *out-shuffle* leaves the top card of the deck unchanged. After an in-shuffle, the original top card becomes the second card from the top. For example:

   $$\mathit{OutShuffle}(\text{A♠ 2♠ 3♠ 4♠ 5♥ 6♥ 7♥ 8♥}) = \text{A♠ 5♥ 2♠ 6♥ 3♠ 7♥ 4♠ 8♥}$$

   $$\mathit{InShuffle}(\text{A♠ 2♠ 3♠ 4♠ 5♥ 6♥ 7♥ 8♥}) = \text{5♥ A♠ 6♥ 2♠ 7♥ 3♠ 8♥ 4♠}$$

   (If you are unfamiliar with playing cards, please refer to the Wikipedia article https://en.wikipedia.org/wiki/Standard_52-card_deck.)

   Suppose we start with a deck of 2^n distinct cards, for some non-negative integer n. What is the effect of performing exactly n perfect in-shuffles on this deck? Prove your answer is correct!

"CS 374" Fall 2014 — Homework 1

Due Tuesday, September 9, 2014 at noon

Groups of up to three students may submit common solutions for each problem in this homework and in all future homeworks. You are responsible for forming you own groups; you are welcome to advertise for group members on Piazza. You need not use the same group for every homework, or even for every problem in a single homework. Please clearly print the names and NetIDs of each of your group members at the top of each submitted solution, along with ***one*** discussion section where we should return your graded work. If you submit hand-written solutions, please use the last three pages of this homework as templates.

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1. Give regular expressions for each of the following languages over the alphabet $\{0, 1\}$. You do not need to prove your answers are correct.

   (a) All strings with an odd number of 1s.

   (b) All strings with at most three 0s.

   (c) All strings that do not contain the substring 010.

   (d) All strings in which every occurrence of the substring 00 occurs before every occurrence of the substring 11.

2. Recall that the ***reversal*** w^R of a string w is defined recursively as follows:

$$w^R = \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ x \bullet a & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

   The reversal L^R of a language L is defined as the set of reversals of all strings in L:

$$L^R := \left\{ w^R \;\middle|\; w \in L \right\}$$

   (a) Prove that $(L^*)^R = (L^R)^*$ for every language L.

   (b) Prove that the reversal of any regular language is also a regular language. (You may assume part (a) even if you haven't proved it yet.)

   You may assume the following facts without proof:

   - $L^* \bullet L^* = L^*$ for every language L.
   - $(w^R)^R = w$ for every string w.
   - $(x \bullet y)^R = y^R \bullet x^R$ for all strings x and y.

   [Hint: Yes, all three proofs use induction, but induction on what? And yes, all ***three*** *proofs.]*

3. Describe context-free grammars for each of the following languages over the alphabet $\{0, 1\}$. Explain *briefly* why your grammars are correct; in particular, describe *in English* the language generated by each non-terminal in your grammars. (We are not looking for full formal proofs of correctness, but convincing evidence that *you* understand why your answers are correct.)

   (a) The set of all strings with more than twice as many 0s as 1s.

   (b) The set of all strings that are *not* palindromes.

   *(c) **[Extra credit]** The set of all strings that are *not* equal to ww for any string w.
   [Hint: $a + b = b + a$.]

“CS 374” Fall 2014 — Homework 2

Due Tuesday, September 16, 2014 at noon

Groups of up to three students may submit common solutions for each problem in this homework and in all future homeworks. You are responsible for forming you own groups; you are welcome to advertise for group members on Piazza. You need not use the same group for every homework, or even for every problem in a single homework. Please clearly print the names and NetIDs of each of your group members at the top of each submitted solution, along with ***one*** discussion section where we should return your graded work. If you submit hand-written solutions, please use the last three pages of this homework as templates.

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1. ***C comments*** are the set of strings over alphabet $\Sigma = \{*, /, \texttt{A}, \diamond, \hookleftarrow\}$ that form a proper comment in the C program language and its descendants, like C++ and Java. Here $\hookleftarrow$ represents the newline character, $\diamond$ represents any other whitespace character (like the space and tab characters), and A represents any non-whitespace character other than $*$ or /.[1] There are two types of C comments:

   - Line comments: Strings of the form // ... $\hookleftarrow$.
   - Block comments: Strings of the form /* ... */.

   Following the C99 standard, we explicitly disallow ***nesting*** comments of the same type. A line comment starts with // and ends at the first $\hookleftarrow$ after the opening //. A block comment starts with /* and ends at the the first */ completely after the opening /*; in particular, every block comment has at least two *s. For example, the following strings are all valid C comments:

   - /***/
   - //◇//◇$\hookleftarrow$
   - /*///◇*◇$\hookleftarrow$**/
   - /*◇//◇$\hookleftarrow$◇*/

   On the other hand, the following strings are *not* valid C comments:

   - /*/
   - //◇//◇$\hookleftarrow$◇$\hookleftarrow$
   - /*◇/*◇*/◇*/

   (a) Describe a DFA that accepts the set of all C comments.

   (b) Describe a DFA that accepts the set of all strings composed entirely of blanks($\diamond$), newlines($\hookleftarrow$), and C comments.

   You must explain *in English* how your DFAs work. Drawings or formal descriptions without English explanations will receive no credit, even if they are correct.

2. Construct a DFA for the following language over alphabet $\{0,1\}$:

$$L = \left\{ w \in \{0,1\}^* \;\middle|\; \begin{array}{l}\text{the number represented by binary string } w \text{ is divisible} \\ \text{by 19, but the length of } w \text{ is not a multiple of 23}\end{array} \right\}.$$

You must explain *in English* how your DFA works. A formal description without an English explanation will receive no credit, even if it is correct. Don't even try to draw the DFA.

3. Prove that each of the following languages is *not* regular.

   (a) $\left\{ w \in \{0\}^* \;\middle|\; \text{length of } w \text{ is a perfect square; that is, } |w| = k^2 \text{ for some integer } k \right\}$.

   (b) $\left\{ w \in \{0,1\}^* \;\middle|\; \text{the number represented by } w \text{ as a binary string is a perfect square} \right\}$.

*4. **[Extra credit]** Suppose L is a regular language which guarantees to contain at least one palindrome. Prove that if an n-state DFA M accepts L, then L contains a palindrome of length polynomial in n. What is the polynomial bound you get?

[1] The actual C commenting syntax is considerably more complex than described here, because of character and string literals.

- The opening `/*` or `//` of a comment must not be inside a string literal (`"..."`) or a (multi-)character literal (`'...'`).
- The opening double-quote of a string literal must not be inside a character literal (`'"'`) or a comment.
- The closing double-quote of a string literal must not be escaped (`\"`)
- The opening single-quote of a character literal must not be inside a string literal (`"...'..."`) or a comment.
- The closing single-quote of a character literal must not be escaped (`\'`)
- A backslash escapes the next symbol if and only if it is not itself escaped (`\\`) or inside a comment.

For example, the string `"/*\\\"*/"/*"/*\"/*"*/` is a valid string literal (representing the 5-character string `/*\"\*/`, which is itself a valid block comment!) followed immediately by a valid block comment. For this homework question, just pretend that the characters `'`, `"`, and `\` don't exist.

The C++ commenting is even more complicated, thanks to the addition of *raw* string literals. Don't ask.

Some C and C++ compilers do support nested block comments, in violation of the language specification. A few other languages, like OCaml, explicitly allow nesting comments.

"CS 374" Fall 2014 — Homework 3

Due Tuesday, September 23, 2014 at noon

- As usual, groups of up to three students may submit common solutions for this assignment. Each group should submit exactly ***one*** solution for each problem. Please clearly print the names and NetIDs of each of your group members at the top of each submitted solution, along with ***one*** discussion section where we should return your graded work. If you submit hand-written solutions, please use the last three pages of this handout as templates.
- If a question asks you to construct an NFA, you are welcome to use ε-transitions.

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1. For each of the following regular expressions, describe or draw two finite-state machines:

   - An NFA that accepts the same language, using Thompson's algorithm (described in class and in the notes)
   - An equivalent DFA, using the incremental subset construction described in class. For each state in your DFA, identify the corresponding subset of states in your NFA. Your DFA should have no unreachable states.

   (a) $(\mathtt{01} + \mathtt{10})^*(\mathtt{0} + \mathtt{1} + \varepsilon)$

   (b) $\mathtt{1}^* + (\mathtt{10})^* + (\mathtt{100})^*$

2. Prove that for any regular language L, the following languages are also regular:

   (a) $\textsc{Substrings}(L) := \{x \mid wxy \in L \text{ for some } w, y \in \Sigma^*\}$

   (b) $\textsc{Half}(L) := \{w \mid ww \in L\}$

   *[Hint: Describe how to transform a DFA for L into NFAs for $\textsc{Substrings}(L)$ and $\textsc{Half}(L)$. What do your NFAs have to guess? Don't forget to explain **in English** how your NFAs work.]*

3. Which of the following languages over the alphabet $\Sigma = \{\mathtt{0}, \mathtt{1}\}$ are regular and which are not? Prove your answers are correct. Recall that Σ^+ denotes the set of all *nonempty* strings over Σ.

   (a) $\left\{wxw \;\middle|\; w, x \in \Sigma^+\right\}$

   (b) $\left\{wxx \;\middle|\; w, x \in \Sigma^+\right\}$

   (c) $\left\{wxwy \;\middle|\; w, x, y \in \Sigma^+\right\}$

   (d) $\left\{wxxy \;\middle|\; w, x, y \in \Sigma^+\right\}$

"CS 374" Fall 2014 — Homework 4

Due Tuesday, October 7, 2014 at noon

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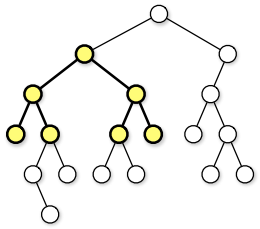
1. For this problem, a *subtree* of a binary tree means any connected subgraph. A binary tree is *complete* if every internal node has two children, and every leaf has exactly the same depth. Describe and analyze a recursive algorithm to compute the *largest complete subtree* of a given binary tree. Your algorithm should return both the root and the depth of this subtree.



The largest complete subtree of this binary tree has depth 2.

2. Consider the following cruel and unusual sorting algorithm.

```
CRUEL(A[1..n]):
  if n > 1
      CRUEL(A[1..n/2])
      CRUEL(A[n/2+1..n])
      UNUSUAL(A[1..n])
```

```
UNUSUAL(A[1..n]):
  if n = 2
      if A[1] > A[2]                          ⟨⟨the only comparison!⟩⟩
          swap A[1] ↔ A[2]
  else
      for i ← 1 to n/4                        ⟨⟨swap 2nd and 3rd quarters⟩⟩
          swap A[i + n/4] ↔ A[i + n/2]
      UNUSUAL(A[1..n/2])                      ⟨⟨recurse on left half⟩⟩
      UNUSUAL(A[n/2+1..n])                    ⟨⟨recurse on right half⟩⟩
      UNUSUAL(A[n/4+1..3n/4])                 ⟨⟨recurse on middle half⟩⟩
```

Notice that the comparisons performed by the algorithm do not depend at all on the values in the input array; such a sorting algorithm is called ***oblivious***. Assume for this problem that the input size n is always a power of 2.

   (a) Prove by induction that CRUEL correctly sorts any input array. *[Hint: Consider an array that contains $n/4$ 1s, $n/4$ 2s, $n/4$ 3s, and $n/4$ 4s. Why is this special case enough?]*

   (b) Prove that CRUEL would *not* correctly sort if we removed the for-loop from UNUSUAL.

   (c) Prove that CRUEL would *not* correctly sort if we swapped the last two lines of UNUSUAL.

   (d) What is the running time of UNUSUAL? Justify your answer.

   (e) What is the running time of CRUEL? Justify your answer.

3. In the early 20th century, a German mathematician developed a variant of the Towers of Hanoi game, which quickly became known in the American literature as "Liberty Towers".[1] In this variant, there is a row of $k \geq 3$ pegs, numbered in order from 1 to k. In a single turn, for any index i, you can move the smallest disk on peg i to either peg $i-1$ or peg $i+1$, subject to the usual restriction that you cannot place a bigger disk on a smaller disk. Your mission is to move a stack of n disks from peg 1 to peg k.

   (a) Describe and analyze a recursive algorithm for the case $k = 3$. ***Exactly*** how many moves does your algorithm perform?

   (b) Describe and analyze a recursive algorithm for the case $k = n+1$ that requires at most $O(n^3)$ moves. To simplify the algorithm, assume that n is a power of 2. *[Hint: Use part (a).]*

   (c) **[Extra credit]** Describe and analyze a recursive algorithm for the case $k = n+1$ that requires at most $O(n^2)$ moves. Do *not* assume that n is a power of 2. *[Hint: Don't use part (a).]*

   (d) **[Extra credit]** Describe and analyze a recursive algorithm for the case $k = \sqrt{n}+1$ that requires at most a polynomial number of moves. To simplify the algorithm, assume that n is a power of 4. What polynomial bound do you get? *[Hint: Use part (a)!]*

   ★(e) **[Extra extra credit]** Describe and analyze a recursive algorithm for arbitrary n and k. How small must k be (as a function of n) so that the number of moves is bounded by a polynomial in n? (This is actually an open research problem, a phrase which here means "Nobody knows the best answer.")

[1] No, not really. During World War I, many German-derived or Germany-related names were changed to more patriotic variants. For example, sauerkraut became "liberty cabbage", hamburgers became "liberty sandwiches", frankfurters became "Liberty sausages" or "hot dogs", German measles became "liberty measles", dachsunds became "liberty pups", German shepherds became "Alsatians", and pinochle (the card game) became "Liberty". For more recent anti-French examples, see "freedom fries", "freedom toast", and "liberty lip lock". Americans are weird.

"CS 374" Fall 2014 — Homework 5

Due Tuesday, October 14, 2014 at noon

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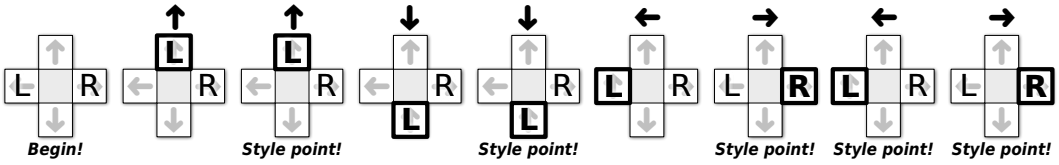
1. ***Dance Dance Revolution*** is a dance video game, first introduced in Japan by Konami in 1998. Players stand on a platform marked with four arrows, pointing forward, back, left, and right, arranged in a cross pattern. During play, the game plays a song and scrolls a sequence of n arrows (←, ↑, ↓, or →) from the bottom to the top of the screen. At the precise moment each arrow reaches the top of the screen, the player must step on the corresponding arrow on the dance platform. (The arrows are timed so that you'll step with the beat of the song.)

   You are playing a variant of this game called "Vogue Vogue Revolution", where the goal is to play perfectly but move as little as possible. When an arrow reaches the top of the screen, if one of your feet is already on the correct arrow, you are awarded one style point for maintaining your current pose. If neither foot is on the right arrow, you must move one (and *only* one) of your feet from its current location to the correct arrow on the platform. If you ever step on the wrong arrow, or fail to step on the correct arrow, or move more than one foot at a time, or move either foot when you are already standing on the correct arrow, or insult Beyoncé, all your style points are immediately taken away and you lose.

   How should you move your feet to maximize your total number of style points? For purposes of this problem, assume you always start with you left foot on ← and you right foot on →, and that you've memorized the entire sequence of arrows. For example, if the sequence is ↑↑↓↓←→←→, you can earn 5 style points by moving you feet as shown below:



   Describe and analyze an efficient algorithm to find the maximum number of style points you can earn during a given VVR routine. Your input is an array $Arrow[1..n]$ containing the sequence of arrows.

2. Recall that a *palindrome* is any string that is exactly the same as its reversal, like `I`, or `DEED`, or `RACECAR`, or `AMANAPLANACATACANALPANAMA`.

   Any string can be decomposed into a sequence of palindrome substrings. For example, the string `BUBBASEESABANANA` ("Bubba sees a banana.") can be broken into palindromes in the following ways (among many others):

   `BUB` • `BASEESAB` • `ANANA`

   `B` • `U` • `BB` • `A` • `SEES` • `ABA` • `NAN` • `A`

   `B` • `U` • `BB` • `A` • `SEES` • `A` • `B` • `ANANA`

   `B` • `U` • `B` • `B` • `A` • `S` • `E` • `E` • `S` • `A` • `B` • `ANA` • `N` • `A`

   Describe and analyze an efficient algorithm to find the smallest number of palindromes that make up a given input string. For example, given the input string `BUBBASEESABANANA`, your algorithm would return the integer 3.

3. Suppose you are given a DFA $M = (\{0,1\}, Q, s, A, \delta)$ and a binary string $w \in \{0,1\}^*$.

   (a) Describe and analyze an algorithm that computes the longest subsequence of w that is accepted by M, or correctly reports that M does not accept any subsequence of w.

   *(b) **[Extra credit]** Describe and analyze an algorithm that computes the *shortest supersequence* of w that is accepted by M, or correctly reports that M does not accept any supersequence of w. (Recall that a string x is a supersequence of w if and only if w is a subsequence of x.)

   Analyze both of your algorithms in terms of the parameters $n = |w|$ and $k = |Q|$.

Rubric (for all dynamic programming problems): As usual, a score of x on the following 10-point scale corresponds to a score of $\lceil x/3 \rceil$ on the 4-point homework scale.

- 6 points for a correct recurrence, described either using mathematical notation or as pseudocode for a recursive algorithm.
  - + 1 point for a clear **English** description of the function you are trying to evaluate. (Otherwise, we don't even know what you're *trying* to do.) **Automatic zero if the English description is missing.**
  - + 1 point for stating how to call your function to get the final answer.
  - + 1 point for base case(s). −½ for one *minor* bug, like a typo or an off-by-one error.
  - + 3 points for recursive case(s). −1 for each *minor* bug, like a typo or an off-by-one error. **No credit for the rest of the problem if the recursive case(s) are incorrect.**
- 4 points for details of the dynamic programming algorithm
  - + 1 point for describing the memoization data structure
  - + 2 points for describing a correct evaluation order; a clear picture is sufficient. If you use nested loops, be sure to specify the nesting order.
  - + 1 point for time analysis
- It is *not* necessary to state a space bound.
- For problems that ask for an algorithm that computes an optimal *structure*—such as a subset, partition, subsequence, or tree—an algorithm that computes only the *value* or *cost* of the optimal structure is sufficient for full credit.
- Official solutions usually include pseudocode for the final iterative dynamic programming algorithm, ***but this is not required for full credit***. If your solution includes iterative pseudocode, you do not need to separately describe the recurrence, data structure, or evaluation order. (But you still need to describe the underlying recursive function in English.)
- Official solutions will provide target time bounds. Algorithms that are faster than this target are worth more points; slower algorithms are worth fewer points, typically by 2 or 3 points for each factor of n. Partial credit is scaled to the new maximum score, and all points above 10 are recorded as extra credit.

  We rarely include these target time bounds in the actual questions, because when we do include them, significantly more students turn in algorithms that meet the target time bound but don't work (earning 0/10) instead of correct algorithms that are slower than the target time bound (earning 8/10).

"CS 374" Fall 2014 — Homework 6

Due Tuesday, October 21, 2014 at noon

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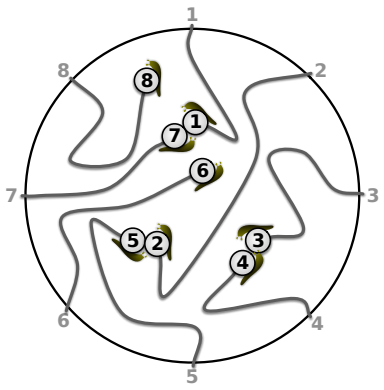
1. Every year, as part of its annual meeting, the Antarctican Snail Lovers of Upper Glacierville hold a Round Table Mating Race. Several high-quality breeding snails are placed at the edge of a round table. The snails are numbered in order around the table from 1 to n. During the race, each snail wanders around the table, leaving a trail of slime behind it. The snails have been specially trained never to fall off the edge of the table or to cross a slime trail, even their own. If two snails meet, they are declared a breeding pair, removed from the table, and whisked away to a romantic hole in the ground to make little baby snails. Note that some snails may never find a mate, even if the race goes on forever.



The end of a typical Antarctican SLUG race. Snails 6 and 8 never find mates.
The organizers must pay $M[3,4] + M[2,5] + M[1,7]$.

For every pair of snails, the Antarctican SLUG race organizers have posted a monetary reward, to be paid to the owners if that pair of snails meets during the Mating Race. Specifically, there is a two-dimensional array $M[1..n, 1..n]$ posted on the wall behind the Round Table, where $M[i, j] = M[j, i]$ is the reward to be paid if snails i and j meet. Rewards may be positive, negative, or zero.

Describe and analyze an algorithm to compute the maximum total reward that the organizers could be forced to pay, given the array M as input.

2. Consider a weighted version of the class scheduling problem, where different classes offer different number of credit hours, which are of course totally unrelated to the duration of the class lectures. Given arrays $S[1..n]$ of start times, an array $F[1..n]$ of finishing times, and an array $H[1..n]$ of credit hours as input, your goal is to choose a set of non-overlapping classes with the largest possible number of credit hours.

   (a) Prove that the greedy algorithm described in class — Choose the class that ends first and recurse — does *not* always return the best schedule.

   (b) Describe an efficient algorithm to compute the best schedule.

In addition to submitting a solution on paper as usual, please *individually* submit an electronic solution for this problem on CrowdGrader. Please see the course web page for detailed instructions.

3. Suppose you have just purchased a new type of hybrid car that uses fuel extremely efficiently, but can only travel 100 miles on a single battery. The car's fuel is stored in a single-use battery, which must be replaced after at most 100 miles. The actual fuel is virtually free, but the batteries are expensive and can only be installed by licensed battery-replacement technicians. Thus, even if you decide to replace your battery early, you must still pay full price for the new battery to be installed. Moreover, because these batteries are in high demand, no one can afford to own more than one battery at a time.

   Suppose you are trying to get from San Francisco to New York City on the new Inter-Continental Super-Highway, which runs in a direct line between these two cities. There are several fueling stations along the way; each station charges a different price for installing a new battery. Before you start your trip, you carefully print the Wikipedia page listing the locations and prices of every fueling station on the ICSH. Given this information, how do you decide the best places to stop for fuel?

   More formally, suppose you are given two arrays $D[1..n]$ and $C[1..n]$, where $D[i]$ is the distance from the start of the highway to the ith station, and $C[i]$ is the cost to replace your battery at the ith station. Assume that your trip starts and ends at fueling stations (so $D[1] = 0$ and $D[n]$ is the total length of your trip), and that your car starts with an empty battery (so you must install a new battery at station 1).

   (a) Describe and analyze a greedy algorithm to find the minimum number of refueling stops needed to complete your trip. Don't forget to prove that your algorithm is correct.

   (b) But what you really want to minimize is the total *cost* of travel. Show that your greedy algorithm in part (a) does *not* produce an optimal solution when extended to this setting.

   (c) Describe an efficient algorithm to compute the locations of the fuel stations you should stop at to minimize the total cost of travel.

"CS 374" Fall 2014 — Homework 7

Due Tuesday, October 28, 2014 at noon

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1. You are standing next to a water pond, and you have three empty jars. Each jar holds a positive integer number of gallons; the capacities of the three jars may or may not be different. You want one of the jars (which one doesn't matter) to contain exactly k gallons of water, for some integer k. You are only allowed to perform the following operations:

   (a) Fill a jar with water from the pond until the jar is full.

   (b) Empty a jar of water by pouring water into the pond.

   (c) Pour water from one jar to another, until either the first jar is empty or the second jar is full, whichever happens first.

   For example, suppose your jars hold 6, 10, and 15 gallons. Then you can put 13 gallons of water into the third jar in six steps:

   - Fill the third jar from the pond.
   - Fill the first jar from the third jar. (Now the third jar holds 9 gallons.)
   - Empty the first jar into the pond.
   - Fill the second jar from the pond.
   - Fill the first jar from the second jar. (Now the second jar holds 4 gallons.)
   - Empty the second jar into the third jar.

   Describe an efficient algorithm that finds the minimum number of operations required to obtain a jar containing exactly k gallons of water, or reports correctly that obtaining exactly k gallons of water is impossible, given the capacities of the three jars and a positive integer k as input. For example, given the four numbers $6, 10, 15$ and 13 as input, your algorithm should return the number 6 (for the sequence of operations listed above).

2. Consider a directed graph G, where each edge is colored either red, white, or blue. A walk[1] in G is called a *French flag walk* if its sequence of edge colors is red, white, blue, red, white, blue, and so on. More formally, a walk $v_0 \to v_1 \to \cdots \to v_k$ is a French flag path if, for every integer i, the edge $v_i \to v_{i+1}$ is red if $i \bmod 3 = 0$, white if $i \bmod 3 = 1$, and blue if $i \bmod 3 = 2$.

   Describe an efficient algorithm to find all vertices in a given edge-colored directed graph G that can be reached from a given vertex v through a French flag walk.

3. Suppose we are given a directed acyclic graph G where every edge e has a positive integer weight $w(e)$, along with two specific vertices s and t and a positive integer W.

   (a) Describe an efficient algorithm to find the *longest* path (meaning the largest number of edges) from s to t in G with total weight at most W. *[Hint: Use dynamic programming.]*

   (b) **[Extra credit]** Solve part (a) with a running time that does not depend on W.

---

[1] Recall that a ***walk*** in G is a sequence of vertices $v_0 \to v_1 \to \cdots \to v_k$, such that $v_{i-1} \to v_i$ is an edge in G for every index i. A ***path*** is a walk in which no vertex appears more than once.

"CS 374" Fall 2014 — Homework 8

Due Tuesday, November 4, 2014 at noon

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1. After a grueling algorithms midterm, you decide to take the bus home. Since you planned ahead, you have a schedule that lists the times and locations of every stop of every bus in Champaign-Urbana. Champaign-Urbana is currently suffering from a plague of zombies, so even though the bus stops have fences that *supposedly* keep the zombies out, you'd still like to spend as little time waiting at bus stops as possible. Unfortunately, there isn't a single bus that visits both your exam building and your home; you must transfer between buses at least once.

   Describe and analyze an algorithm to determine a sequence of bus rides from Siebel to your home, that minimizes the total time you spend waiting at bus stops. You can assume that there are b different bus lines, and each bus stops n times per day. Assume that the buses run exactly on schedule, that you have an accurate watch, and that walking between bus stops is too dangerous to even contemplate.

2. It is well known that the global economic collapse of 2017 was caused by computer scientists indiscriminately abusing weaknesses in the currency exchange market. *Arbitrage* was a money-making scheme that takes advantage of inconsistencies in currency exchange rates. Suppose a currency trader with $1,000,000 discovered that 1 US dollar could be traded for 120 Japanese yen, 1 yen could be traded for 0.01 euros, and 1 euro could be traded for 1.2 US dollars. Then by converting his money from dollars to yen, then from yen to euros, and finally from euros back to dollars, the trader could instantly turn his $1,000,000 into $1,440,000! The cycle of currencies $\$ \to ¥ \to € \to \$$ was called an ***arbitrage cycle***. Finding and exploiting arbitrage cycles before the prices were corrected required extremely fast algorithms. Of course, now that the entire world uses plastic bags as currency, such abuse is impossible.

   Suppose n different currencies are traded in the global currency market. You are given a two-dimensional array $Exch[1..n, 1..n]$ of exchange rates between every pair of currencies; for all indices i and j, one unit of currency i buys $Exch[i, j]$ units of currency j. (Do *not* assume that $Exch[i, j] \cdot Exch[j, i] = 1$.)

   (a) Describe an algorithm that computes an array $Most[1..n]$, where $Most[i]$ is the largest amount of currency i that you can obtain by trading, starting with one unit of currency 1, assuming there are no arbitrage cycles.

   (b) Describe an algorithm to determine whether the given array of currency exchange rates creates an arbitrage cycle.

3. Describe and analyze and algorithm to find the *second smallest spanning tree* of a given undirected graph G with weighted edges, that is, the spanning tree of G with smallest total weight except for the minimum spanning tree. Because the minimum spanning tree is haunted, or something.

"CS 374" Fall 2014 — Homework 9

Due Tuesday, November 18, 2014 at noon

The following questions ask you to describe various Turing machines. In each problem, give *both* a formal description of your Turing machine in terms of specific states, tape symbols, and transition functions *and* explain in English how your Turing machine works. In particular:

- Clearly specify what variant of Turing machine you are using: Number of tapes, number of heads, allowed head motions, halting conditions, and so on.
- Include the type signature of your machine's transition function. The standard model uses a transition function whose signature is $\delta : Q \times \Gamma \to Q \times \Gamma \times \{-1, +1\}$.
- If necessary, break your Turing machine into smaller functional pieces, and describe those pieces separately (both formally and in English).
- Use state names that convey their meaning/purpose.

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1. Describe a Turing machine that computes the function $\lceil \log_2 n \rceil$. Given the string 1^n as input, for any positive integer n, your machine should return the string $1^{\lceil \log_2 n \rceil}$ as output. For example, given the input string 1111111111111 (thirteen 1s), your machine should output the string 1111, because $2^3 < 13 \leq 2^4$.

2. A ***binary-tree*** Turing machine uses an infinite binary tree as its tape; that is, *every* cell in the tape has a left child and a right child. At each step, the head moves from its current cell to its *P*arent, its *L*eft child, or to its *R*ight child. Thus, the transition function of such a machine has the form $\delta : Q \times \Gamma \to Q \times \Gamma \times \{P, L, R\}$. The input string is initially given along the left spine of the tape.

   Prove that any binary-tree Turing machine can be simulated by a standard Turing machine. That is, given any binary-tree Turing machine $M = (\Gamma, \square, \Sigma, Q, \mathsf{start}, \mathsf{accept}, \mathsf{reject}, \delta)$, describe a standard Turing machine $M' = (\Gamma', \square', \Sigma, Q', \mathsf{start}', \mathsf{accept}', \mathsf{reject}', \delta')$ that accepts and rejects exactly the same input strings as M. Be sure to describe how a single transition of M is simulated by M'.

   In addition to submitting paper solutions, please also electronically submit your solution to this problem on CrowdGrader.

3. **[Extra credit]**

   A ***tag***-Turing machine has two heads: one can only read, the other can only write. Initially, the read head is located at the left end of the tape, and the write head is located at the first blank after the input string. At each transition, the read head can either move one cell to the right or stay put, but the write head *must* write a symbol to its current cell and move one cell to the right. Neither head can ever move to the left.

   Prove that any standard Turing machine can be simulated by a tag-Turing machine. That is, given any standard Turing machine M, formally describe a tag-Turing machine M' that accepts and rejects exactly the same strings as M. Be sure to describe how a single transition of M is simulated by M'.

"CS 374" Fall 2014 — Homework 10

Due Tuesday, December 2, 2014 at noon

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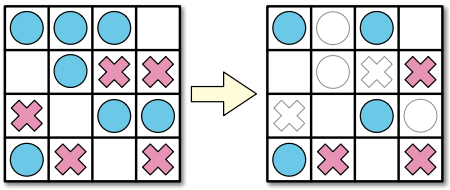
1. Consider the following problem, called BOXDEPTH: Given a set of n axis-aligned rectangles in the plane, how big is the largest subset of these rectangles that contain a common point?

   (a) Describe a polynomial-time reduction from BOXDEPTH to MAXCLIQUE.

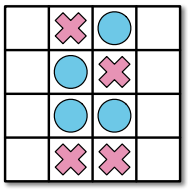
   (b) Describe and analyze a polynomial-time algorithm for BOXDEPTH. *[Hint: Don't try to optimize the running time; $O(n^3)$ is good enough.]*

   (c) Why don't these two results imply that P=NP?

2. Consider the following solitaire game. The puzzle consists of an $n \times m$ grid of squares, where each square may be empty, occupied by a red stone, or occupied by a blue stone. The goal of the puzzle is to remove some of the given stones so that the remaining stones satisfy two conditions: (1) every row contains at least one stone, and (2) no column contains stones of both colors. For some initial configurations of stones, reaching this goal is impossible.



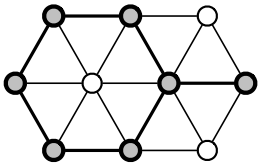
A solvable puzzle and one of its many solutions.



An unsolvable puzzle.

   Prove that it is NP-hard to determine, given an initial configuration of red and blue stones, whether this puzzle can be solved.

3. A subset S of vertices in an undirected graph G is called ***triangle-free*** if, for every triple of vertices $u, v, w \in S$, at least one of the three edges uv, uw, vw is *absent* from G. Prove that finding the size of the largest triangle-free subset of vertices in a given undirected graph is NP-hard.



A triangle-free subset of 7 vertices.
This is ***not*** the largest triangle-free subset in this graph.

   In addition to submitting paper solutions, please also electronically submit your solution to this problem on CrowdGrader.

4. **[Extra credit]** Describe a direct polynomial-time reduction from 4COLOR to 3COLOR. (This is significantly harder than the opposite direction, which you'll see in lab on Wednesday. Don't go through the Cook-Levin Theorem.)

"CS 374" Fall 2014 ✧ Homework 11

Due Tuesday, December 9, 2014 at noon

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1. Recall that w^R denotes the reversal of string w; for example, $\texttt{TURING}^R = \texttt{GNIRUT}$. Prove that the following language is undecidable.

$$\textsc{RevAccept} := \{\langle M \rangle \mid M \text{ accepts } \langle M \rangle^R\}$$

2. Let M be a Turing machine, let w be an arbitrary input string, and let s be an integer. We say that ***M accepts w in space s*** if, given w as input, M accesses only the first s cells on the tape and eventually accepts.

   (a) Prove that the following language is decidable:

$$\{\langle M, w \rangle \mid M \text{ accepts } w \text{ in space } |w|^2\}$$

   (b) Prove that the following language is undecidable:

$$\{\langle M \rangle \mid M \text{ accepts at least one string } w \text{ in space } |w|^2\}$$

3. ***[Extra credit]*** For each of the following languages, either prove that the language is decidable, or prove that the language is undecidable.

   (a) $L_0 = \{\langle M \rangle \mid \text{given any input string, } M \text{ eventually leaves its } \texttt{start} \text{ state}\}$

   (b) $L_1 = \{\langle M \rangle \mid M \text{ decides } L_0\}$

   (c) $L_2 = \{\langle M \rangle \mid M \text{ decides } L_1\}$

   (d) $L_3 = \{\langle M \rangle \mid M \text{ decides } L_2\}$

   (e) $L_4 = \{\langle M \rangle \mid M \text{ decides } L_3\}$

1. Prove that every non-negative integer can be represented as the sum of distinct powers of 2. ("Write it in binary" is not a proof; it's just a restatement of what you have to prove.)

2. Suppose you and your 8-year-old cousin Elmo decide to play a game with a rectangular bar of chocolate, which has been scored into an $n \times m$ grid of squares. You and Elmo alternate turns. On each turn, you or Elmo choose one of the available pieces of chocolate and break it along one of the grid lines into two smaller rectangles. Thus, at all times, each piece of chocolate is an $a \times b$ rectangle for some positive integers a and b; in particular, a 1×1 piece cannot be broken into smaller pieces. The game ends when all the pieces are individual squares. The winner is the player who breaks the last piece.

   Describe a strategy for winning this game. When should you take the first move, and when should you offer it to Elmo? On each turn, how do you decide which piece to break and where? Prove your answers are correct. *[Hint: Let's play a 3×3 game. You go first. Oh, and I'm kinda busy right now, so could you just play for me whenever it's my turn? Thanks.]*

3. **[To think about later]** Now consider a variant of the previous chocolate-bar game, where on each turn you can *either* break a piece into two smaller pieces *or eat a* 1×1 *piece*. This game ends when all the chocolate is gone. The winner is the player who eats the last bite of chocolate (*not* the player who eats the *most* chocolate). Describe a strategy for winning this game, and prove that your strategy works.

These lab problems ask you to prove some simple claims about recursively-defined string functions and concatenation. In each case, we want a self-contained proof by induction that relies on the formal recursive definitions, *not* on intuition. In particular, your proofs must refer to the formal recursive definition of string concatenation:

$$w \bullet z := \begin{cases} z & \text{if } w = \varepsilon \\ a \cdot (x \bullet z) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

You may also use any of the following facts, which we proved in class:

Lemma 1: Concatenating nothing does nothing: For every string w, we have $w \bullet \varepsilon = w$.

Lemma 2: Concatenation adds length: $|w \bullet x| = |w| + |x|$ for all strings w and x.

Lemma 3: Concatenation is associative: $(w \bullet x) \bullet y = w \bullet (x \bullet y)$ for all strings w, x, and y.

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1. Let $\#(a, w)$ denote the number of times symbol a appears in string w; for example,

$$\#(\texttt{0}, \texttt{000010101010010100}) = 12 \qquad \text{and} \qquad \#(\texttt{1}, \texttt{000010101010010100}) = 6.$$

   (a) Give a formal recursive definition of $\#(a, w)$.

   (b) Prove by induction that $\#(a, w \bullet z) = \#(a, w) + \#(a, z)$ for any symbol a and any strings w and z.

2. The ***reversal*** w^R of a string w is defined recursively as follows:

$$w^R := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ x^R \bullet a & \text{if } w = a \cdot x \end{cases}$$

   (a) Prove that $(w \bullet x)^R = x^R \bullet w^R$ for all strings w and x.

   (b) Prove that $(w^R)^R = w$ for every string w.

Give regular expressions that describe each of the following languages over the alphabet {`0`, `1`}. We won't get to all of these in section.

---

1. All strings containing at least three `0`s.

2. All strings containing at least two `0`s and at least one `1`.

3. All strings containing the substring `000`.

4. All strings *not* containing the substring `000`.

5. All strings in which every run of `0`s has length at least 3.

6. All strings such that every substring `000` appears after every `1`.

7. Every string except `000`. *[Hint: Don't try to be clever.]*

8. All strings w such that *in every prefix of* w, the number of `0`s and `1`s differ by at most 1.

*9. All strings w such that *in every prefix of* w, the number of `0`s and `1`s differ by at most 2.

★10. All strings in which the substring `000` appears an even number of times.
(For example, `0001000` and `0000` are in this language, but `00000` is not.)

Jeff showed the context-free grammars in class on Tuesday; in each example, the grammar itself is on the left; the explanation for each non-terminal is on the right.

- Properly nested strings of parentheses.

  $$S \to \varepsilon \mid S\texttt{(}S\texttt{)} \qquad \text{properly nested parentheses}$$

  Here is a different grammar for the same language:

  $$S \to \varepsilon \mid \texttt{(}S\texttt{)} \mid SS \qquad \text{properly nested parentheses}$$

- $\{\texttt{0}^m\texttt{1}^n \mid m \neq n\}$. This is the set of all binary strings composed of some number of `0`s followed by a different number of `1`s.

$S \to A \mid B$	all strings $\texttt{0}^m\texttt{1}^n$ where $m \neq n$
$A \to \texttt{0}A \mid \texttt{0}C$	all strings $\texttt{0}^m\texttt{1}^n$ where $m > n$
$B \to B\texttt{1} \mid C\texttt{1}$	all strings $\texttt{0}^m\texttt{1}^n$ where $m < m$
$C \to \varepsilon \mid \texttt{0}C\texttt{1}$	all strings $\texttt{0}^n\texttt{1}^n$ for some integer n

---

Give context-free grammars for each of the following languages. For each grammar, describe *in English* the language for each non-terminal, and in the examples above. As usual, we won't get to all of these in section.

1. Binary palindromes: Strings over $\{\texttt{0}, \texttt{1}\}$ that are equal to their reversals. For example: `00111100` and `0100010`, but not `01100`.

2. $\{\texttt{0}^{2n}\texttt{1}^n \mid n \geq 0\}$

3. $\{\texttt{0}^m\texttt{1}^n \mid m \neq 2n\}$

4. $\{\texttt{0}, \texttt{1}\}^* \setminus \{\texttt{0}^{2n}\texttt{1}^n \mid n \geq 0\}$

5. Strings of properly nested parentheses `()`, brackets `[]`, and braces `{}`. For example, the string `([]){}` is in this language, but the string `([)]` is not, because the left and right delimiters don't match.

6. Strings over $\{\texttt{0}, \texttt{1}\}$ where the number of `0`s is equal to the number of `1`s.

7. Strings over $\{\texttt{0}, \texttt{1}\}$ where the number of `0`s is *not* equal to the number of `1`s.

Construct DFA that accept each of the following languages over the alphabet {`0`, `1`}. We won't get to all of these in section.

---

1. (a) $(0+1)^*$
   (b) $\emptyset$
   (c) $\{\epsilon\}$

2. Every string except `000`.

3. All strings containing the substring `000`.

4. All strings *not* containing the substring `000`.

5. All strings in which the reverse of the string is the binary representation of a integer divisible by 3.

6. All strings w such that *in every prefix of* w, the number of `0`s and `1`s differ by at most 2.

Prove that each of the following languages is not regular.

1. Binary palindromes: Strings over $\{\mathtt{0}, \mathtt{1}\}$ that are equal to their reversals. For example: `00111100` and `0100010`, but not `01100`. *[Hint: We did this in class.]*

2. $\{\mathtt{0}^{2n}\mathtt{1}^n \mid n \geq 0\}$

3. $\{\mathtt{0}^m\mathtt{1}^n \mid m \neq 2n\}$

4. Strings over $\{\mathtt{0}, \mathtt{1}\}$ where the number of `0`s is exactly twice the number of `1`s.

5. Strings of properly nested parentheses `()`, brackets `[]`, and braces `{}`. For example, the string `([]){}` is in this language, but the string `([)]` is not, because the left and right delimiters don't match.

6. $\left\{\mathtt{0}^{2^n} \;\middle|\; n \geq 0\right\}$ — Strings of `0`s whose length is a power of 2.

7. Strings of the form $w_1\#w_2\#\cdots\#w_n$ for some $n \geq 2$, where each substring w_i is a string in $\{\mathtt{0}, \mathtt{1}\}^*$, and some pair of substrings w_i and w_j are equal.

For each of the following languages over the alphabet $\Sigma = \{\mathtt{0}, \mathtt{1}\}$, either prove the language is regular (by giving an equivalent regular expression, DFA, or NFA) or prove that the language is not regular (using a fooling set argument). Exactly half of these languages are regular.

1. $\{\mathtt{0}^n\mathtt{1}\mathtt{0}^n \mid n \geq 0\}$

2. $\{\mathtt{0}^n\mathtt{1}\mathtt{0}^n w \mid n \geq 0 \text{ and } w \in \Sigma^*\}$

3. $\{w\mathtt{0}^n\mathtt{1}\mathtt{0}^n x \mid w \in \Sigma^* \text{ and } n \geq 0 \text{ and } x \in \Sigma^*\}$

4. Strings in which the number of **0**s and the number of **1**s differ by at most 2.

5. Strings such that *in every prefix*, the number of **0**s and the number of **1**s differ by at most 2.

6. Strings such that *in every substring*, the number of **0**s and the number of **1**s differ by at most 2.

Here are several problems that are easy to solve in $O(n)$ time, essentially by brute force. Your task is to design algorithms for these problems that are significantly faster.

1. (a) Suppose $A[1..n]$ is an array of n distinct integers, sorted so that $A[1] < A[2] < \cdots < A[n]$. Each integer $A[i]$ could be positive, negative, or zero. Describe a fast algorithm that either computes an index i such that $A[i] = i$ or correctly reports that no such index exists..

   (b) Now suppose $A[1..n]$ is a sorted array of n distinct ***positive*** integers. Describe an even faster algorithm that either computes an index i such that $A[i] = i$ or correctly reports that no such index exists. *[Hint: This is **really** easy.]*

2. Suppose we are given an array $A[1..n]$ such that $A[1] \geq A[2]$ and $A[n-1] \leq A[n]$. We say that an element $A[x]$ is a ***local minimum*** if both $A[x-1] \geq A[x]$ and $A[x] \leq A[x+1]$. For example, there are exactly six local minima in the following array:

9	**7**	7	2	**1**	3	7	5	**4**	7	**3**	**3**	4	8	**6**	9
	▲			▲				▲		▲	▲			▲	

   Describe and analyze a fast algorithm that returns the index of one local minimum. For example, given the array above, your algorithm could return the integer 5, because $A[5]$ is a local minimum. *[Hint: With the given boundary conditions, any array **must** contain at least one local minimum. Why?]*

3. (a) Suppose you are given two sorted arrays $A[1..n]$ and $B[1..n]$ containing distinct integers. Describe a fast algorithm to find the median (meaning the nth smallest element) of the union $A \cup B$. For example, given the input

   $$A[1..8] = [0, 1, 6, 9, 12, 13, 18, 20] \qquad B[1..8] = [2, 4, 5, 8, 17, 19, 21, 23]$$

   your algorithm should return the integer 9. *[Hint: What can you learn by comparing one element of A with one element of B?]*

   (b) ***To think about on your own:*** Now suppose you are given two sorted arrays $A[1..m]$ and $B[1..n]$ and an integer k. Describe a fast algorithm to find the kth smallest element in the union $A \cup B$. For example, given the input

   $$A[1..8] = [0, 1, 6, 9, 12, 13, 18, 20] \qquad B[1..5] = [2, 5, 7, 17, 19] \qquad k = 6$$

   your algorithm should return the integer 7.

Here are several problems that are easy to solve in $O(n)$ time, essentially by brute force. Your task is to design algorithms for these problems that are significantly faster, and ***prove*** that your algorithm is correct.

1. (a) Suppose $A[1..n]$ is an array of n distinct integers, sorted so that $A[1] < A[2] < \cdots < A[n]$. Each integer $A[i]$ could be positive, negative, or zero. Describe a fast algorithm that either computes an index i such that $A[i] = i$ or correctly reports that no such index exists..

   (b) Now suppose $A[1..n]$ is a sorted array of n distinct ***positive*** integers. Describe an even faster algorithm that either computes an index i such that $A[i] = i$ or correctly reports that no such index exists. *[Hint: This is **really** easy.]*

2. Suppose we are given an array $A[1..n]$ such that $A[1] \geq A[2]$ and $A[n-1] \leq A[n]$. We say that an element $A[x]$ is a ***local minimum*** if both $A[x-1] \geq A[x]$ and $A[x] \leq A[x+1]$. For example, there are exactly six local minima in the following array:

9	**7**	7	2	**1**	3	7	5	**4**	7	**3**	**3**	4	8	**6**	9
	▲			▲				▲		▲	▲			▲	

   Describe and analyze a fast algorithm that returns the index of one local minimum. For example, given the array above, your algorithm could return the integer 5, because $A[5]$ is a local minimum. *[Hint: With the given boundary conditions, any array **must** contain at least one local minimum. Why?]*

3. (a) Suppose you are given two sorted arrays $A[1..n]$ and $B[1..n]$ containing distinct integers. Describe a fast algorithm to find the median (meaning the nth smallest element) of the union $A \cup B$. For example, given the input

   $$A[1..8] = [0, 1, 6, 9, 12, 13, 18, 20] \qquad B[1..8] = [2, 4, 5, 8, 17, 19, 21, 23]$$

   your algorithm should return the integer 9. *[Hint: What can you learn by comparing one element of A with one element of B?]*

   (b) ***To think about on your own:*** Now suppose you are given two sorted arrays $A[1..m]$ and $B[1..n]$ and an integer k. Describe a fast algorithm to find the kth smallest element in the union $A \cup B$. For example, given the input

   $$A[1..8] = [0, 1, 6, 9, 12, 13, 18, 20] \qquad B[1..5] = [2, 5, 7, 17, 19] \qquad k = 6$$

   your algorithm should return the integer 7.

A ***subsequence*** of a sequence (for example, an array, linked list, or string), obtained by removing zero or more elements and keeping the rest in the same sequence order. A subsequence is called a ***substring*** if its elements are contiguous in the original sequence. For example:

- `SUBSEQUENCE`, `UBSEQU`, and the empty string ε are all substrings of the string `SUBSEQUENCE`;
- `SBSQNC`, `UEQUE`, and `EEE` are all subsequences of `SUBSEQUENCE` but not substrings;
- `QUEUE`, `SSS`, and `FOOBAR` are not subsequences of `SUBSEQUENCE`.

Describe recursive backtracking algorithms for the following problems. *Don't worry about running times.*

1. Given an array $A[1..n]$ of integers, compute the length of a ***longest increasing subsequence***. A sequence $B[1..\ell]$ is *increasing* if $B[i] > B[i-1]$ for every index $i \geq 2$. For example, given the array

   $$\langle 3, \underline{1}, \underline{4}, 1, \underline{5}, 9, 2, \underline{6}, 5, 3, 5, \underline{8}, \underline{9}, 7, 9, 3, 2, 3, 8, 4, 6, 2, 7 \rangle$$

   your algorithm should return the integer 6, because $\langle 1, 4, 5, 6, 8, 9 \rangle$ is a longest increasing subsequence (one of many).

2. Given an array $A[1..n]$ of integers, compute the length of a ***longest decreasing subsequence***. A sequence $B[1..\ell]$ is *decreasing* if $B[i] < B[i-1]$ for every index $i \geq 2$. For example, given the array

   $$\langle 3, 1, 4, 1, 5, \underline{9}, 2, \underline{6}, 5, 3, \underline{5}, 8, 9, 7, 9, 3, 2, 3, 8, \underline{4}, 6, \underline{2}, 7 \rangle$$

   your algorithm should return the integer 5, because $\langle 9, 6, 5, 4, 2 \rangle$ is a longest decreasing subsequence (one of many).

3. Given an array $A[1..n]$ of integers, compute the length of a ***longest alternating subsequence***. A sequence $B[1..\ell]$ is *alternating* if $B[i] < B[i-1]$ for every even index $i \geq 2$, and $B[i] > B[i-1]$ for every odd index $i \geq 3$. For example, given the array

   $$\langle \underline{3}, \underline{1}, \underline{4}, \underline{1}, \underline{5}, 9, \underline{2}, \underline{6}, \underline{5}, 3, 5, \underline{8}, 9, \underline{7}, \underline{9}, \underline{3}, 2, 3, \underline{8}, \underline{4}, \underline{6}, \underline{2}, \underline{7} \rangle$$

   your algorithm should return the integer 17, because $\langle 3, 1, 4, 1, 5, 2, 6, 5, 8, 7, 9, 3, 8, 4, 6, 2, 7 \rangle$ is a longest alternating subsequence (one of many).

A ***subsequence*** of a sequence (for example, an array, a linked list, or a string), obtained by removing zero or more elements and keeping the rest in the same sequence order. A subsequence is called a ***substring*** if its elements are contiguous in the original sequence. For example:

- `SUBSEQUENCE`, `UBSEQU`, and the empty string ε are all substrings of the string `SUBSEQUENCE`;
- `SBSQNC`, `UEQUE`, and `EEE` are all subsequences of `SUBSEQUENCE` but not substrings;
- `QUEUE`, `SSS`, and `FOOBAR` are not subsequences of `SUBSEQUENCE`.

Describe and analyze ***dynamic programming*** algorithms for the following problems. For the first three, use the backtracking algorithms you developed on Wednesday.

1. Given an array $A[1..n]$ of integers, compute the length of a longest ***increasing*** subsequence of A. A sequence $B[1..\ell]$ is *increasing* if $B[i] > B[i-1]$ for every index $i \geq 2$.

2. Given an array $A[1..n]$ of integers, compute the length of a longest ***decreasing*** subsequence of A. A sequence $B[1..\ell]$ is *decreasing* if $B[i] < B[i-1]$ for every index $i \geq 2$.

3. Given an array $A[1..n]$ of integers, compute the length of a longest ***alternating*** subsequence of A. A sequence $B[1..\ell]$ is *alternating* if $B[i] < B[i-1]$ for every even index $i \geq 2$, and $B[i] > B[i-1]$ for every odd index $i \geq 3$.

4. Given an array $A[1..n]$ of integers, compute the length of a longest ***convex*** subsequence of A. A sequence $B[1..\ell]$ is *convex* if $B[i]-B[i-1] > B[i-1]-B[i-2]$ for every index $i \geq 3$.

5. Given an array $A[1..n]$, compute the length of a longest ***palindrome*** subsequence of A. Recall that a sequence $B[1..\ell]$ is a *palindrome* if $B[i] = B[\ell - i + 1]$ for every index i.

Basic steps in developing a dynamic programming algorithm

1. **Formulate the problem recursively.** This is the hard part. There are two distinct but equally important things to include in your formulation.

   (a) **Specification.** First, give a clear and precise English description of the problem you are claiming to solve. Don't describe *how* to solve the problem at this stage; just describe *what* the problem actually is. Otherwise, the reader has no way to know what your recursive algorithm is *supposed* to compute.

   (b) **Solution.** Second, give a clear recursive formula or algorithm for the whole problem in terms of the answers to smaller instances of *exactly* the same problem. It generally helps to think in terms of a recursive definition of your inputs and outputs. If you discover that you need a solution to a *similar* problem, or a slightly *related* problem, you're attacking the wrong problem; go back to step 1.

2. **Build solutions to your recurrence from the bottom up.** Write an algorithm that starts with the base cases of your recurrence and works its way up to the final solution, by considering intermediate subproblems in the correct order. This stage can be broken down into several smaller, relatively mechanical steps:

   (a) **Identify the subproblems.** What are all the different ways can your recursive algorithm call itself, starting with some initial input? For example, the argument to RECFIBO is always an integer between 0 and n.

   (b) **Analyze space and running time.** The number of possible distinct subproblems determines the space complexity of your memoized algorithm. To compute the time complexity, add up the running times of all possible subproblems, *ignoring the recursive calls*. For example, if we already know F_{i-1} and F_{i-2}, we can compute F_i in $O(1)$ time, so computing the first n Fibonacci numbers takes $O(n)$ time.

   (c) **Choose a data structure to memoize intermediate results.** For most problems, each recursive subproblem can be identified by a few integers, so you can use a multidimensional array. For some problems, however, a more complicated data structure is required.

   (d) **Identify dependencies between subproblems.** Except for the base cases, every recursive subproblem depends on other subproblems—which ones? Draw a picture of your data structure, pick a generic element, and draw arrows from each of the other elements it depends on. Then formalize your picture.

   (e) **Find a good evaluation order.** Order the subproblems so that each subproblem comes *after* the subproblems it depends on. Typically, this means you should consider the base cases first, then the subproblems that depends only on base cases, and so on. More formally, the dependencies you identified in the previous step define a partial order over the subproblems; in this step, you need to find a linear extension of that partial order. ***Be careful!***

   (f) **Write down the algorithm.** You know what order to consider the subproblems, and you know how to solve each subproblem. So do that! If your data structure is an array, this usually means writing a few nested for-loops around your original recurrence.

1. It's almost time to show off your flippin' sweet dancing skills! Tomorrow is the big dance contest you've been training for your entire life, except for that summer you spent with your uncle in Alaska hunting wolverines. You've obtained an advance copy of the the list of n songs that the judges will play during the contest, in chronological order.

   You know all the songs, all the judges, and your own dancing ability extremely well. For each integer k, you know that if you dance to the kth song on the schedule, you will be awarded exactly $Score[k]$ points, but then you will be physically unable to dance for the next $Wait[k]$ songs (that is, you cannot dance to songs $k+1$ through $k+Wait[k]$). The dancer with the highest total score at the end of the night wins the contest, so you want your total score to be as high as possible.

   Describe and analyze an efficient algorithm to compute the maximum total score you can achieve. The input to your sweet algorithm is the pair of arrays $Score[1..n]$ and $Wait[1..n]$.

2. A *shuffle* of two strings X and Y is formed by interspersing the characters into a new string, keeping the characters of X and Y in the same order. For example, the string **BANANAANANAS** is a shuffle of the strings **BANANA** and **ANANAS** in several different ways.

   BANANA$_{\textbf{ANANAS}}$ **BAN**$_{\textbf{ANA}}$**ANA**$_{\textbf{NAS}}$ **B**$_{\textbf{AN}}$**AN**$_{\textbf{A}}$**A**$_{\textbf{NA}}$**NA**$_{\textbf{S}}$

   Similarly, the strings **PRODGYRNAMAMMIINCG** and **DYPRONGARMAMMICING** are both shuffles of **DYNAMIC** and **PROGRAMMING**:

   PRO$^{\textbf{D}}$**G**$^{\textbf{Y}}$**R**$^{\textbf{NAM}}$**AMMI**$^{\textbf{I}}$**N**$^{\textbf{C}}$**G** **DY**$_{\textbf{PRO}}$**N**$_{\textbf{G}}$**A**$_{\textbf{R}}$**M**$_{\textbf{AMM}}$**IC**$_{\textbf{ING}}$

   Describe and analyze an efficient algorithm to determine, given three strings $A[1..m]$, $B[1..n]$, and $C[1..m+n]$, whether C is a shuffle of A and B.

Basic steps in developing a dynamic programming algorithm

1. **Formulate the problem recursively.** This is the hard part. There are two distinct but equally important things to include in your formulation.

   (a) **Specification.** First, give a clear and precise English description of the problem you are claiming to solve. Don't describe *how* to solve the problem at this stage; just describe *what* the problem actually is. Otherwise, the reader has no way to know what your recursive algorithm is *supposed* to compute.

   (b) **Solution.** Second, give a clear recursive formula or algorithm for the whole problem in terms of the answers to smaller instances of *exactly* the same problem. It generally helps to think in terms of a recursive definition of your inputs and outputs. If you discover that you need a solution to a *similar* problem, or a slightly *related* problem, you're attacking the wrong problem; go back to step 1.

2. **Build solutions to your recurrence from the bottom up.** Write an algorithm that starts with the base cases of your recurrence and works its way up to the final solution, by considering intermediate subproblems in the correct order. This stage can be broken down into several smaller, relatively mechanical steps:

   (a) **Identify the subproblems.** What are all the different ways can your recursive algorithm call itself, starting with some initial input? For example, the argument to RECFIBO is always an integer between 0 and n.

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   (c) **Choose a data structure to memoize intermediate results.** For most problems, each recursive subproblem can be identified by a few integers, so you can use a multidimensional array. For some problems, however, a more complicated data structure is required.

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   (f) **Write down the algorithm.** You know what order to consider the subproblems, and you know how to solve each subproblem. So do that! If your data structure is an array, this usually means writing a few nested for-loops around your original recurrence.

1. Suppose you are given a sequence of non-negative integers separated by + and × signs; for example:

$$2 \times 3 + 0 \times 6 \times 1 + 4 \times 2$$

You can change the value of this expression by adding parentheses in different places. For example:

$$2 \times (3 + (0 \times (6 \times (1 + (4 \times 2))))) = 6$$
$$(((((2 \times 3) + 0) \times 6) \times 1) + 4) \times 2 = 80$$
$$((2 \times 3) + (0 \times 6)) \times (1 + (4 \times 2)) = 108$$
$$(((2 \times 3) + 0) \times 6) \times ((1 + 4) \times 2) = 360$$

Describe and analyze an algorithm to compute, given a list of integers separated by + and × signs, the largest possible value we can obtain by inserting parentheses.

Your input is an array $A[0..2n]$ where each $A[i]$ is an integer if i is even and + or × if i is odd. Assume any arithmetic operation in your algorithm takes $O(1)$ time.

Basic steps in developing a dynamic programming algorithm

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   (f) **Write down the algorithm.** You know what order to consider the subproblems, and you know how to solve each subproblem. So do that! If your data structure is an array, this usually means writing a few nested for-loops around your original recurrence.

Recall the class scheduling problem described in lecture on Tuesday. We are given two arrays $S[1..n]$ and $F[1..n]$, where $S[i] < F[i]$ for each i, representing the start and finish times of n classes. Your goal is to find the largest number of classes you can take without ever taking two classes simultaneously. We showed in class that the following greedy algorithm constructs an optimal schedule:

> Choose the course that *ends first*, discard all conflicting classes, and recurse.

But this is not the only greedy strategy we could have tried. For each of the following alternative greedy algorithms, either prove that the algorithm always constructs an optimal schedule, or describe a small input example for which the algorithm does not produce an optimal schedule. Assume that all algorithms break ties arbitrarily (that is, in a manner that is completely out of your control).

[Hint: Exactly three of these greedy strategies actually work.]

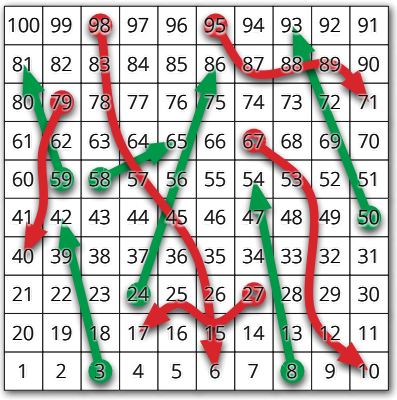
1. Choose the course x that *ends last*, discard classes that conflict with x, and recurse.
2. Choose the course x that *starts first*, discard all classes that conflict with x, and recurse.
3. Choose the course x that *starts last*, discard all classes that conflict with x, and recurse.
4. Choose the course x with *shortest duration*, discard all classes that conflict with x, and recurse.
5. Choose a course x that *conflicts with the fewest other courses*, discard all classes that conflict with x, and recurse.
6. If no classes conflict, choose them all. Otherwise, discard the course with *longest duration* and recurse.
7. If no classes conflict, choose them all. Otherwise, discard a course that *conflicts with the most other courses* and recurse.
8. Let x be the class with the *earliest start time*, and let y be the class with the *second earliest start time*.
   - If x and y are disjoint, choose x and recurse on everything but x.
   - If x completely contains y, discard x and recurse.
   - Otherwise, discard y and recurse.
9. If any course x completely contains another course, discard x and recurse. Otherwise, choose the course y that *ends last*, discard all classes that conflict with y, and recurse.

For each of the problems below, transform the input into a graph and apply a standard graph algorithm that you've seen in class. Whenever you use a standard graph algorithm, you ***must*** provide the following information. (I recommend actually using a bulleted list.)

- What are the vertices?
- What are the edges? Are they directed or undirected?
- If the vertices and/or edges have associated values, what are they?
- What problem do you need to solve on this graph?
- What standard algorithm are you using to solve that problem?
- What is the running time of your entire algorithm, *including* the time to build the graph, *as a function of the original input parameters*?

---

1. ***Snakes and Ladders*** is a classic board game, originating in India no later than the 16th century. The board consists of an $n \times n$ grid of squares, numbered consecutively from 1 to n^2, starting in the bottom left corner and proceeding row by row from bottom to top, with rows alternating to the left and right. Certain pairs of squares, always in different rows, are connected by either "snakes" (leading down) or "ladders" (leading up). Each square can be an endpoint of at most one snake or ladder.



A typical Snakes and Ladders board.
Upward straight arrows are ladders; downward wavy arrows are snakes.

   You start with a token in cell 1, in the bottom left corner. In each move, you advance your token up to k positions, for some fixed constant k (typically 6). If the token ends the move at the *top* end of a snake, you ***must*** slide the token down to the bottom of that snake. If the token ends the move at the *bottom* end of a ladder, you ***may*** move the token up to the top of that ladder.

   Describe and analyze an algorithm to compute the smallest number of moves required for the token to reach the last square of the grid.

2. Let G be a connected undirected graph. Suppose we start with two coins on two arbitrarily chosen vertices of G. At every step, each coin *must* move to an adjacent vertex. Describe and analyze an algorithm to compute the minimum number of steps to reach a configuration where both coins are on the same vertex, or to report correctly that no such configuration is reachable. The input to your algorithm consists of a graph $G = (V, E)$ and two vertices $u, v \in V$ (which may or may not be distinct).

For each of the problems below, transform the input into a graph and apply a standard graph algorithm that you've seen in class. Whenever you use a standard graph algorithm, you ***must*** provide the following information. (I recommend actually using a bulleted list.)

- What are the vertices?
- What are the edges? Are they directed or undirected?
- If the vertices and/or edges have associated values, what are they?
- What problem do you need to solve on this graph?
- What standard algorithm are you using to solve that problem?
- What is the running time of your entire algorithm, *including* the time to build the graph, *as a function of the original input parameters*?

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1. Inspired by the previous lab, you decided to organize a Snakes and Ladders competition with n participants. In this competition, each game of Snakes and Ladders involves three players. After the game is finished, they are ranked first, second and third. Each player may be involved in any (non-negative) number of games, and the number needs not be equal among players.

   At the end of the competition, m games have been played. You realized that you had forgotten to implement a proper rating system, and therefore decided to produce the overall ranking of all n players as you see fit. However, to avoid being too suspicious, if player A ranked better than player B in any game, then A must rank better than B in the overall ranking.

   You are given the list of players involved and the ranking in each of the m games. Describe and analyze an algorithm to produce an overall ranking of the n players that satisfies the condition, or correctly reports that it is impossible.

2. There are n galaxies connected by m intergalactic teleport-ways. Each teleport-way joins two galaxies and can be traversed in both directions. Also, each teleport-way e has an associated toll of c_e dollars, where c_e is a positive integer. A teleport-way can be used multiple times, but the toll must be paid every time it is used.

   Judy wants to travel from galaxy u to galaxy v, but teleportation is not very pleasant and she would like to minimize the number of times she needs to teleport. However, she wants the total cost to be a multiple of five dollars, because carrying small bills is not pleasant either.

   (a) Describe and analyze an algorithm to compute the smallest number of times Judy needs to teleport to travel from galaxy u to galaxy v while the total cost is a multiple of five dollars.

   (b) Solve (a), but now assume that Judy has a coupon that allows her to waive the toll once.

Suppose we are given both an undirected graph G with weighted edges and a minimum spanning tree T of G.

1. Describe an efficient algorithm to update the minimum spanning tree when the weight of one edge $e \in T$ is decreased.

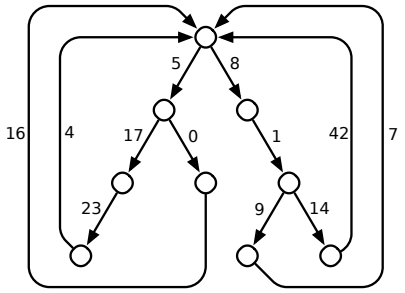
2. Describe an efficient algorithm to update the minimum spanning tree when the weight of one edge $e \notin T$ is increased.

3. Describe an efficient algorithm to update the minimum spanning tree when the weight of one edge $e \in T$ is increased.

4. Describe an efficient algorithm to update the minimum spanning tree when the weight of one edge $e \notin T$ is decreased.

In all cases, the input to your algorithm is the edge e and its new weight; your algorithms should modify T so that it is still a minimum spanning tree. Of course, we could just recompute the minimum spanning tree from scratch in $O(E \log V)$ time, but you can do better.

1. A *looped tree* is a weighted, directed graph built from a binary tree by adding an edge from every leaf back to the root. Every edge has non-negative weight.



A looped tree.

   (a) How much time would Dijkstra's algorithm require to compute the shortest path between two vertices u and v in a looped tree with n nodes?

   (b) Describe and analyze a faster algorithm.

2. After graduating you accept a job with Aerophobes-Я-Us, the leading traveling agency for people who hate to fly. Your job is to build a system to help customers plan airplane trips from one city to another. All of your customers are afraid of flying (and by extension, airports), so any trip you plan needs to be as short as possible. You know all the departure and arrival times of all the flights on the planet.

   Suppose one of your customers wants to fly from city X to city Y. Describe an algorithm to find a sequence of flights that minimizes the *total time in transit*—the length of time from the initial departure to the final arrival, including time at intermediate airports waiting for connecting flights. *[Hint: Build an appropriate graph from the input data and apply Dijkstra's algorithm.]*

Describe Turing machines that compute the following functions.

In particular, specify the transition functions $\delta : Q \times \Gamma \to Q \times \Gamma \times \{-1, +1\}$ for each machine either by writing out a table or by drawing a graph. Recall that $\delta(p, \$) = (q, @, +1)$ means that if the Turing machine is in state p and reads the symbol $\$$ from the tape, then it will change to state q, write the symbol $@$ to the tape, and move one step to the right. In a *drawing* of a Turing machine, this transition is indicated by an edge from p to q with the label "$\$/@, +1$".

Give your states short mnemonic names that suggest their purpose. Naming your states well won't just make it easier to understand; it will also make it easier to design.

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1. DOUBLE: Given a string $w \in \{0, 1\}^*$ as input, return the string ww as output.

2. POWER: Given a string of the form 1^n as input, return the string 1^{2^n} as output.

Describe how to simulate an arbitrary Turing machine to make it ***error-tolerant***. Specifically, given an arbitrary Turing machine M, describe a new Turing machine M' that accepts and rejects exactly the same strings as M, even though an evil pixie named Lenny will move the head of M' to an ***arbitrary*** location on the tape some finite number of *unknown* times during the execution of M'.

You do not have to describe M' in complete detail, but do give enough details that a seasoned Turing machine programmer could work out the remaining mechanical details.

As stated, this problem has no solution! If M halts on all inputs after a finite number of steps, then Lenny can make any substring of the input string completely invisible to M. For example, if the true input string is `INPUT-STRING`, Lenny can make M believe the input string is actually `IMPING`, by moving the head to the second `I` whenever it tries to move to `R`, and by moving the head to `P` when it tries to move to `U`. Because M halts after a finite number of steps, Lenny only has a finite number of opportunities to move the head.

In fact, with more care, Lenny can make M think the input string is *any* string that uses only symbols from the actual input string; if the true input string is `INPUT-STRING`, Lenny can make M believe the input string is actually `GRINNING-PUTIN-IS-GRINNING`.)

However, there are several different ways to rescue the problem. For each of the following restrictions on Lenny's behavior, and for any Turing machine M, one can design a Turing machine M' that simulates M despite Lenny's interference.

- Lenny can move the head only a *bounded* number of times. For example: Lenny can move the head at most 374 times.
- Whenever Lenny moves the head, he changes the state of the machine to a special error state `lenny`.
- Whenever Lenny moves the head, he moves it to the left end of the tape.
- Whenever Lenny moves the head, he moves it to a blank cell to the right of all non-blank cells.
- Whenever Lenny moves the head, he moves it to a cell containing a particular symbol in the input alphabet, say `0`.

Describe algorithms for the following problems. The input for each problem is string $\langle M, w \rangle$ that encodes a standard (one-tape, one-track, one-head) Turing machine M whose tape alphabet is $\{0, 1, \square\}$ and a string $w \in \{0, 1\}^*$.

1. Does M accept w after at most $|w|^2$ steps?
2. If we run M with input w, does M ever move its head to the right?

2½. If we run M with input w, does M ever move its head to the right twice in a row?

2¾. If we run M with input w, does M move its head to the right more than $2^{|w|}$ times?

3. If we run M with input w, does M ever change a symbol on the tape?

3½. If we run M with input w, does M ever change a $\square$ on the tape to either 0 or 1?

4. If we run M with input w, does M ever leave its start state?

In contrast, as we will see later, the following problems are all undecidable!

1. Does M accept w?

1½. If we run M with input w, does M ever halt?

2. If we run M with input w, does M ever move its head to the right three times in a row?
3. If we run M with input w, does M ever change a $\square$ on the tape to 1?

3½. If we run M with input w, does M ever change either 0 or 1 on the tape to $\square$?

4. If we run M with input w, does M ever reenter its start state?

1. Suppose you are given a magic black box that somehow answers the following decision problem in *polynomial time*:

   - INPUT: A boolean circuit K with n inputs and one output .
   - OUTPUT: TRUE if there are input values $x_1, x_2, \ldots, x_n \in \{\text{TRUE}, \text{FALSE}\}$ that make K output TRUE, and FALSE otherwise.

   Using this black box as a subroutine, describe an algorithm that solves the following related search problem *in polynomial time*:

   - INPUT: A boolean circuit K with n inputs and one output.
   - OUTPUT: Input values $x_1, x_2, \ldots, x_n \in \{\text{TRUE}, \text{FALSE}\}$ that make K output TRUE, or NONE if there are no such inputs.

   [Hint: You can use the magic box more than once.]

2. Formally, ***valid 3-coloring*** of a graph $G = (V, E)$ is a function $c : V \to \{1, 2, 3\}$ such that $c(u) \neq c(v)$ for all $uv \in E$. Less formally, a valid 3-coloring assigns each vertex a color, which is either red, green, or blue, such that the endpoints of every edge have different colors.

   Suppose you are given a magic black box that somehow answers the following problem *in polynomial time*:

   - INPUT: An undirected graph G.
   - OUTPUT: TRUE if G has a valid 3-coloring, and FALSE otherwise.

   Using this black box as a subroutine, describe an algorithm that solves the ***3-coloring problem*** *in polynomial time*:

   - INPUT: An undirected graph G.
   - OUTPUT: A valid 3-coloring of G, or NONE if there is no such coloring.

   [Hint: You can use the magic box more than once. The input to the magic box is a graph and ***only*** *a graph, meaning* ***only*** *vertices and edges.]*

Proving that a problem X is NP-hard requires several steps:

- Choose a problem Y that you already know is NP-hard.
- Describe an algorithm to solve Y, using an algorithm for X as a subroutine. Typically this algorithm has the following form: Given an instance of Y, transform it into an instance of X, and then call the magic black-box algorithm for X.
- Prove that your algorithm is correct. This almost always requires two separate steps:
  - Prove that your algorithm transforms "good" instances of Y into "good" instances of X.
  - Prove that your algorithm transforms "bad" instances of Y into "bad" instances of X. Equivalently: Prove that if your transformation produces a "good" instance of X, then it was given a "good" instance of Y.
- Argue that your algorithm for Y runs in polynomial time.

---

1. Recall the following kColor problem: Given an undirected graph G, can its vertices be colored with k colors, so that every edge touches vertices with two different colors?

   (a) Describe a direct polynomial-time reduction from 3Color to 4Color.

   (b) Prove that kColor problem is NP-hard for any $k \geq 3$.

2. Recall that a *Hamiltonian cycle* in a graph G is a cycle that goes through every vertex of G exactly once. Now, a ***tonian cycle*** in a graph G is a cycle that goes through at least *half* of the vertices of G, and a ***Hamilhamiltonian circuit*** in a graph G is a closed walk that goes through every vertex in G exactly *twice*.

   (a) Prove that it is NP-hard to determine whether a given graph contains a tonian cycle.

   (b) Prove that it is NP-hard to determine whether a given graph contains a Hamilhamiltonian circuit.

Proving that a problem X is NP-hard requires several steps:

- Choose a problem Y that you already know is NP-hard.
- Describe an algorithm to solve Y, using an algorithm for X as a subroutine. Typically this algorithm has the following form: Given an instance of Y, transform it into an instance of X, and then call the magic black-box algorithm for X.
- Prove that your algorithm is correct. This almost always requires two separate steps:
  - Prove that your algorithm transforms "good" instances of Y into "good" instances of X.
  - Prove that your algorithm transforms "bad" instances of Y into "bad" instances of X. Equivalently: Prove that if your transformation produces a "good" instance of X, then it was given a "good" instance of Y.
- Argue that your algorithm for Y runs in polynomial time.

Recall that a *Hamiltonian cycle* in a graph G is a cycle that visits every vertex of G exactly once.

1. In class on Thursday, Jeff proved that it is NP-hard to determine whether a given *directed* graph contains a Hamiltonian cycle. Prove that it is NP-hard to determine whether a given *undirected* graph contains a Hamiltonian cycle.

2. A *double Hamiltonian circuit* in a graph G is a closed walk that goes through every vertex in G exactly *twice*. Prove that it is NP-hard to determine whether a given *undirected* graph contains a double Hamiltonian circuit.

Proving that a language L is undecidable by reduction requires several steps:

- Choose a language L' that you already know is undecidable. Typical choices for L' include:

$$\begin{aligned}
\textsc{Accept} &:= \{\langle M, w\rangle \mid M \text{ accepts } w\} \\
\textsc{Reject} &:= \{\langle M, w\rangle \mid M \text{ rejects } w\} \\
\textsc{Halt} &:= \{\langle M, w\rangle \mid M \text{ halts on } w\} \\
\textsc{Diverge} &:= \{\langle M, w\rangle \mid M \text{ diverges on } w\} \\
\textsc{NeverAccept} &:= \{\langle M\rangle \mid \textsc{Accept}(M) = \emptyset\} \\
\textsc{NeverReject} &:= \{\langle M\rangle \mid \textsc{Reject}(M) = \emptyset\} \\
\textsc{NeverHalt} &:= \{\langle M\rangle \mid \textsc{Halt}(M) = \emptyset\} \\
\textsc{NeverDiverge} &:= \{\langle M\rangle \mid \textsc{Diverge}(M) = \emptyset\}
\end{aligned}$$

- Describe an algorithm (really a Turing machine) M' that decides L', using a Turing machine M that decides L as a black box. Typically this algorithm has the following form:

  Given a string w, transform it into another string x,
  such that M accepts x if and only if $w \in L'$.

- Prove that your Turing machine is correct. This almost always requires two separate steps:
  - Prove that if M accepts w then $w \in L'$.
  - Prove that if M rejects w then $w \notin L'$.

Prove that the following languages are undecidable:

1. $\textsc{AcceptIllini} := \{\langle M\rangle \mid M \text{ accepts the string } \texttt{ILLINI}\}$
2. $\textsc{AcceptThree} := \{\langle M\rangle \mid M \text{ accepts exactly three strings}\}$
3. $\textsc{AcceptPalindrome} := \{\langle M\rangle \mid M \text{ accepts at least one palindrome}\}$

Prove that the following languages are undecidable *using Rice's Theorem*:

Rice's Theorem. *Let $\mathcal{X}$ be any nonempty proper subset of the set of acceptable languages. The language* $\textsc{AcceptIn}\mathcal{X} := \left\{ \langle M \rangle \;\middle|\; \textsc{Accept}(M) \in \mathcal{X} \right\}$ *is undecidable.*

1. $\textsc{AcceptRegular} := \left\{ \langle M \rangle \;\middle|\; \textsc{Accept}(M) \text{ is regular} \right\}$
2. $\textsc{AcceptIllini} := \left\{ \langle M \rangle \;\middle|\; M \text{ accepts the string } \texttt{ILLINI} \right\}$
3. $\textsc{AcceptPalindrome} := \left\{ \langle M \rangle \;\middle|\; M \text{ accepts at least one palindrome} \right\}$
4. $\textsc{AcceptThree} := \left\{ \langle M \rangle \;\middle|\; M \text{ accepts exactly three strings} \right\}$
5. $\textsc{AcceptUndecidable} := \left\{ \langle M \rangle \;\middle|\; \textsc{Accept}(M) \text{ is undecidable} \right\}$

To think about later. Which of the following languages are undecidable? How do you prove it?

1. $\textsc{Accept}\{\{\varepsilon\}\} := \left\{ \langle M \rangle \;\middle|\; M \text{ only accepts the string } \varepsilon, \text{ i.e. } \textsc{Accept}(M) = \{\varepsilon\} \right\}$
2. $\textsc{Accept}\{\emptyset\} := \left\{ \langle M \rangle \;\middle|\; M \text{ does not accept any strings, i.e. } \textsc{Accept}(M) = \emptyset \right\}$
3. $\textsc{Accept}\emptyset := \left\{ \langle M \rangle \;\middle|\; \textsc{Accept}(M) \text{ is not an acceptable language} \right\}$
4. $\textsc{Accept}{=}\textsc{Reject} := \left\{ \langle M \rangle \;\middle|\; \textsc{Accept}(M) = \textsc{Reject}(M) \right\}$
5. $\textsc{Accept}{\neq}\textsc{Reject} := \left\{ \langle M \rangle \;\middle|\; \textsc{Accept}(M) \neq \textsc{Reject}(M) \right\}$
6. $\textsc{Accept}{\cup}\textsc{Reject} := \left\{ \langle M \rangle \;\middle|\; \textsc{Accept}(M) \cup \textsc{Reject}(M) = \Sigma^* \right\}$

Write your answers in the separate answer booklet.
Please return this question sheet and your cheat sheet with your answers.

1. For each statement below, check "True" if the statement is ***always*** true and "False" otherwise. Each correct answer is worth +1 point; each incorrect answer is worth −½ point; checking "I don't know" is worth +¼ point; and flipping a coin is (on average) worth +¼ point. You do ***not*** need to prove your answer is correct.

   Read each statement *very* carefully. Some of these are deliberately subtle.

   (a) If $2 + 2 = 5$, then Jeff is the Queen of England.

   (b) For all languages L_1 and L_2, the language $L_1 \cup L_2$ is regular.

   (c) For all languages $L \subseteq \Sigma^*$, if L is not regular, then $\Sigma^* \setminus L$ cannot be represented by a regular expression.

   (d) For all languages L_1 and L_2, if $L_1 \subseteq L_2$ and L_1 is regular, then L_2 is regular.

   (e) For all languages L_1 and L_2, if $L_1 \subseteq L_2$ and L_1 is not regular, then L_2 is not regular.

   (f) For all languages L, if L is regular, then L has no infinite fooling set.

   (g) The language $\left\{ \texttt{0}^m\texttt{1}^n \;\middle|\; 0 \le m + n \le 374 \right\}$ is regular.

   (h) The language $\left\{ \texttt{0}^m\texttt{1}^n \;\middle|\; 0 \le m - n \le 374 \right\}$ is regular.

   (i) For every language L, if the language $L^R = \left\{ w^R \;\middle|\; w \in L \right\}$ is regular, then L is also regular. (Here w^R denotes the reversal of string w; for example, $(\texttt{BACKWARD})^R = \texttt{DRAWKCAB}$.)

   (j) Every context-free language is regular.

2. Let L be the set of strings in $\{\texttt{0}, \texttt{1}\}^*$ in which every run of consecutive `0`s has even length and every run of consecutive `1`s has odd length.

   (a) Give a regular expression that represents L.

   (b) Construct a DFA that recognizes L.

   You do ***not*** need to prove that your answers are correct.

3. For each of the following languages over the alphabet $\{\texttt{0}, \texttt{1}\}$, either ***prove*** that the language is regular or ***prove*** that the language is not regular. ***Exactly one of these two languages is regular.***

   (a) The set of all strings in which the substrings `00` and `11` appear the same number of times.

   (b) The set of all strings in which the substrings `01` and `10` appear the same number of times.

   For example, both of these languages contain the string `1100001101101`.

4. Consider the following recursive function:

$$stutter(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ aa \cdot stutter(x) & \text{if } w = ax \text{ for some } a \in \Sigma \text{ and } x \in \Sigma^* \end{cases}$$

For example, $stutter(\texttt{00101}) = \texttt{0000110011}$.

Prove that for any regular language L, the following languages are also regular.

(a) $\textsc{Stutter}(L) := \{stutter(w) \mid w \in L\}$.

(b) $\textsc{Stutter}^{-1}(L) := \{w \mid stutter(w) \in L\}$.

5. Recall that string concatenation and string reversal are formally defined as follows:

$$w \bullet y := \begin{cases} y & \text{if } w = \varepsilon \\ a \cdot (x \bullet y) & \text{if } w = ax \text{ for some } a \in \Sigma \text{ and } x \in \Sigma^* \end{cases}$$

$$w^R := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ x^R \bullet a & \text{if } w = ax \text{ for some } a \in \Sigma \text{ and } x \in \Sigma^* \end{cases}$$

Prove that $(w \bullet x)^R = x^R \bullet w^R$, for all strings w and x. Your proof should be complete, concise, formal, and self-contained.

Write your answers in the separate answer booklet.
Please return this question sheet and your cheat sheet with your answers.

1. For each statement below, check "True" if the statement is ***always*** true and "False" otherwise. Each correct answer is worth +1 point; each incorrect answer is worth −½ point; checking "I don't know" is worth +¼ point; and flipping a coin is (on average) worth +¼ point. You do ***not*** need to prove your answer is correct.

   Read each statement *very* carefully. Some of these are deliberately subtle.

   (a) If $2 + 2 = 5$, then Jeff is not the Queen of England.

   (b) For all languages L, the language L^* is regular.

   (c) For all languages $L \subseteq \Sigma^*$, if L is can be represented by a regular expression, then $\Sigma^* \setminus L$ can also be represented by a regular expression.

   (d) For all languages L_1 and L_2, if L_2 is regular and $L_1 \subseteq L_2$, then L_1 is regular.

   (e) For all languages L_1 and L_2, if L_2 is not regular and $L_1 \subseteq L_2$, then L_1 is not regular.

   (f) For all languages L, if L is not regular, then every fooling set for L is infinite.

   (g) The language $\left\{ \texttt{0}^m\texttt{10}^n \;\middle|\; 0 \le n - m \le 374 \right\}$ is regular.

   (h) The language $\left\{ \texttt{0}^m\texttt{10}^n \;\middle|\; 0 \le n + m \le 374 \right\}$ is regular.

   (i) For every language L, if L is not regular, then the language $L^R = \left\{ w^R \;\middle|\; w \in L \right\}$ is also not regular. (Here w^R denotes the reversal of string w; for example, $(\texttt{BACKWARD})^R = \texttt{DRAWKCAB}$.)

   (j) Every context-free language is regular.

2. Let L be the set of strings in $\{\texttt{0}, \texttt{1}\}^*$ in which every run of consecutive `0`s has odd length and the total number of `1`s is even.

   For example, the string `11110000010111000` is in L, because it has eight `1`s and three runs of consecutive `0`s, with lengths 5, 1, and 3.

   (a) Give a regular expression that represents L.

   (b) Construct a DFA that recognizes L.

   You do ***not*** need to prove that your answers are correct.

3. For each of the following languages over the alphabet $\{\texttt{0}, \texttt{1}\}$, either ***prove*** that the language is regular or ***prove*** that the language is not regular. ***Exactly one of these two languages is regular.***

   (a) The set of all strings in which the substrings `10` and `01` appear the same number of times.

   (b) The set of all strings in which the substrings `00` and `01` appear the same number of times.

   For example, both of these languages contain the string `1100001101101`.

4. Consider the following recursive function:

$$odds(w) := \begin{cases} w & \text{if } |w| \le 1 \\ a \cdot odds(x) & \text{if } w = abx \text{ for some } a, b \in \Sigma \text{ and } x \in \Sigma^* \end{cases}$$

Intuitively, *odds* removes every other symbol from the input string, starting with the second symbol. For example, $odds(\texttt{0101110}) = \texttt{0010}$.

Prove that for any regular language L, the following languages are also regular.

(a) $\textsc{Odds}(L) := \{odds(w) \mid w \in L\}$.

(b) $\textsc{Odds}^{-1}(L) := \{w \mid odds(w) \in L\}$.

5. Recall that string concatenation and string reversal are formally defined as follows:

$$w \bullet y := \begin{cases} y & \text{if } w = \varepsilon \\ a \cdot (x \bullet y) & \text{if } w = ax \text{ for some } a \in \Sigma \text{ and } x \in \Sigma^* \end{cases}$$

$$w^R := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ x^R \bullet a & \text{if } w = ax \text{ for some } a \in \Sigma \text{ and } x \in \Sigma^* \end{cases}$$

Prove that $(w \bullet x)^R = x^R \bullet w^R$, for all strings w and x. Your proof should be complete, concise, formal, and self-contained. You may assume the following identities, which we proved in class:

- $w \bullet (x \bullet y) = (w \bullet x) \bullet y$ for all strings w, x, and y.
- $|w \bullet x| = |w| + |x|$ for all strings w and x.

Write your answers in the separate answer booklet.
Please return this question sheet and your cheat sheet with your answers.

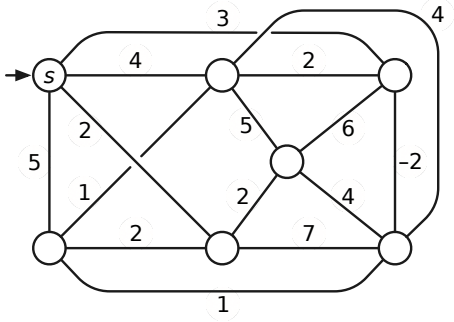
1. ***Clearly*** indicate the edges of the following spanning trees of the weighted graph pictured below. (Pretend that the person grading your exam has bad eyesight.) Some of these subproblems have more than one correct answer. Yes, that edge on the right has negative weight.

   (a) A depth-first spanning tree rooted at s

   (b) A breadth-first spanning tree rooted at s

   (c) A shortest-path tree rooted at s

   (d) A minimum spanning tree



2. An array $A[0..n-1]$ of n distinct numbers is ***bitonic*** if there are unique indices i and j such that $A[(i-1) \bmod n] < A[i] > A[(i+1) \bmod n]$ and $A[(j-1) \bmod n] > A[j] < A[(j+1) \bmod n]$. In other words, a bitonic sequence either consists of an increasing sequence followed by a decreasing sequence, or can be circularly shifted to become so. For example,

4	6	9	8	7	5	1	2	3

   is bitonic, but

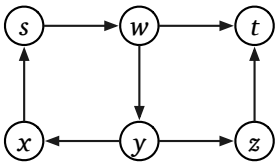
3	6	9	8	7	5	1	2	4

   is *not* bitonic.

   Describe and analyze an algorithm to find the index of the *smallest* element in a given bitonic array $A[0..n-1]$ in $O(\log n)$ time. You may assume that the numbers in the input array are distinct. For example, given the first array above, your algorithm should return 6, because $A[6] = 1$ is the smallest element in that array.

3. Suppose you are given a directed graph $G = (V, E)$ and two vertices s and t. Describe and analyze an algorithm to determine if there is a walk in G from s to t (possibly repeating vertices and/or edges) whose length is divisible by 3.

   For example, given the graph below, with the indicated vertices s and t, your algorithm should return TRUE, because the walk $s \to w \to y \to x \to s \to w \to t$ has length 6.



   [Hint: Build a (different) graph.]

4. The new swap-puzzle game *Candy Swap Saga XIII* involves n cute animals numbered 1 through n. Each animal holds one of three types of candy: circus peanuts, Heath bars, and Cioccolateria Gardini chocolate truffles. You also have a candy in your hand; at the start of the game, you have a circus peanut.

   To earn points, you visit each of the animals in order from 1 to n. For each animal, you can either keep the candy in your hand or exchange it with the candy the animal is holding.

   - If you swap your candy for another candy of the *same* type, you earn one point.
   - If you swap your candy for a candy of a *different* type, you lose one point. (Yes, your score can be negative.)
   - If you visit an animal and decide not to swap candy, your score does not change.

   You *must* visit the animals in order, and once you visit an animal, you can never visit it again.

   Describe and analyze an efficient algorithm to compute your maximum possible score. Your input is an array $C[1..n]$, where $C[i]$ is the type of candy that the ith animal is holding.

5. Let G be a directed graph with weighted edges, and let s be a vertex of G. Suppose every vertex $v \neq s$ stores a pointer $pred(v)$ to another vertex in G. Describe and analyze an algorithm to determine whether these predecessor pointers correctly define a single-source shortest path tree rooted at s. Do ***not*** assume that G has no negative cycles.

Write your answers in the separate answer booklet.
Please return this question sheet and your cheat sheet with your answers.

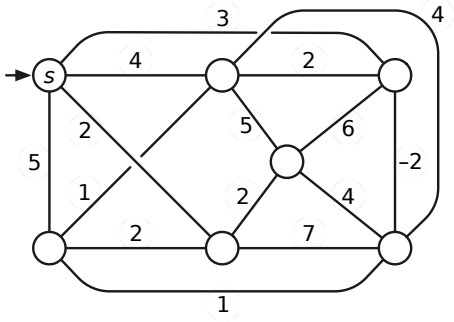
1. ***Clearly*** indicate the edges of the following spanning trees of the weighted graph pictured below. (Pretend that the person grading your exam has bad eyesight.) Some of these subproblems have more than one correct answer. Yes, that edge on the right has negative weight.

   (a) A depth-first spanning tree rooted at s

   (b) A breadth-first spanning tree rooted at s

   (c) A shortest-path tree rooted at s

   (d) A minimum spanning tree



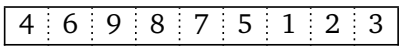
2. Farmers Boggis, Bunce, and Bean have set up an obstacle course for Mr. Fox. The course consists of a row of n booths, each with an integer painted on the front with bright red paint, which could be positive, negative, or zero. Let $A[i]$ denote the number painted on the front of the ith booth. Everyone has agreed to the following rules:

   - At each booth, Mr. Fox ***must*** say either "Ring!" or "Ding!".
   - If Mr. Fox says "Ring!" at the ith booth, he earns a reward of $A[i]$ chickens. (If $A[i] < 0$, Mr. Fox pays a penalty of $-A[i]$ chickens.)
   - If Mr. Fox says "Ding!" at the ith booth, he pays a penalty of $A[i]$ chickens. (If $A[i] < 0$, Mr. Fox earns a reward of $-A[i]$ chickens.)
   - Mr. Fox is forbidden to say the same word more than three times in a row. For example, if he says "Ring!" at booths 6, 7, and 8, then he *must* say "Ding!" at booth 9.
   - All accounts will be settled at the end; Mr. Fox does not actually have to carry chickens through the obstacle course.
   - If Mr. Fox violates any of the rules, or if he ends the obstacle course owing the farmers chickens, the farmers will shoot him.

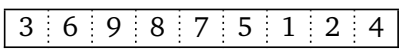
   Describe and analyze an algorithm to compute the largest number of chickens that Mr. Fox can earn by running the obstacle course, given the array $A[1..n]$ of booth numbers as input.

3. Let G be a directed graph with weighted edges, and let s be a vertex of G. Suppose every vertex $v \neq s$ stores a pointer $pred(v)$ to another vertex in G. Describe and analyze an algorithm to determine whether these predecessor pointers correctly define a single-source shortest path tree rooted at s. Do ***not*** assume that G has no negative cycles.

4. An array $A[0..n-1]$ of n distinct numbers is ***bitonic*** if there are unique indices i and j such that $A[(i-1) \bmod n] < A[i] > A[(i+1) \bmod n]$ and $A[(j-1) \bmod n] > A[j] < A[(j+1) \bmod n]$. In other words, a bitonic sequence either consists of an increasing sequence followed by a decreasing sequence, or can be circularly shifted to become so. For example,



is bitonic, but



is *not* bitonic.

Describe and analyze an algorithm to find the index of the *smallest* element in a given bitonic array $A[0..n-1]$ in $O(\log n)$ time. You may assume that the numbers in the input array are distinct. For example, given the first array above, your algorithm should return 6, because $A[6] = 1$ is the smallest element in that array.

5. Suppose we are given an undirected graph G in which every *vertex* has a positive weight.

   (a) Describe and analyze an algorithm to find a *spanning tree* of G with minimum total weight. (The total weight of a spanning tree is the sum of the weights of its vertices.)

   (b) Describe and analyze an algorithm to find a *path* in G from one given vertex s to another given vertex t with minimum total weight. (The total weight of a path is the sum of the weights of its vertices.)

"CS 374": Algorithms and Models of Computation, Fall 2014

Final Exam — Version A — December 16, 2014

Name:	
NetID:	
Section:	1 2 3

#	1	2	3	4	5	6	Total
Score							
Max	20	10	10	10	10	10	70
Grader							

- ***Don't panic!***
- Please print your name and your NetID and circle your discussion section in the boxes above.
- This is a closed-book, closed-notes, closed-electronics exam. If you brought anything except your writing implements and your two double-sided 8½" × 11" cheat sheets, please put it away for the duration of the exam. In particular, you may not use *any* electronic devices.
- **Please read the entire exam before writing anything.** Please ask for clarification if any question is unclear.
- **You have 180 minutes.**
- If you run out of space for an answer, continue on the back of the page, or on the blank pages at the end of this booklet, **but please tell us where to look.** Alternatively, feel free to tear out the blank pages and use them as scratch paper.
- **Please return your cheat sheets and all scratch paper with your answer booklet.**
- If you use a greedy algorithm, you must prove that it is correct to receive credit. **Otherwise, proofs are required only if we specifically ask for them.**
- As usual, answering any (sub)problem with "I don't know" (and nothing else) is worth 25% partial credit. **Yes, even for problem 1.** Correct, complete, but suboptimal solutions are *always* worth more than 25%. A blank answer is not the same as "I don't know".
- ***Good luck!*** And have a great winter break!

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1. For each of the following questions, indicate ***every*** correct answer by marking the "Yes" box, and indicate ***every*** incorrect answer by marking the "No" box. **Assume P $\neq$ NP.** If there is any other ambiguity or uncertainty, mark the "No" box. For example:

[X] Yes	[ ] No	$2+2=4$
[ ] Yes	[X] No	$x+y=5$
[ ] Yes	[X] No	3SAT can be solved in polynomial time.
[X] Yes	[ ] No	Jeff is not the Queen of England.

There are 40 yes/no choices altogether, each worth ½ point.

(a) Which of the following statements is true for ***every*** language $L \subseteq \{0,1\}^*$?

[ ] Yes	[ ] No	L is non-empty.
[ ] Yes	[ ] No	L is decidable or L is infinite (or both).
[ ] Yes	[ ] No	L is accepted by some DFA with 42 states if and only if L is accepted by some NFA with 42 states.
[ ] Yes	[ ] No	If L is regular, then $L \in$ NP.
[ ] Yes	[ ] No	L is decidable if and only if its complement $\overline{L}$ is undecidable.

(b) Which of the following computational models can be simulated by a deterministic Turing machine with three read/write heads, with at most polynomial slow-down in time, assuming P $\neq$ NP?

[ ] Yes	[ ] No	A Java program
[ ] Yes	[ ] No	A deterministic Turing machine with one head
[ ] Yes	[ ] No	A deterministic Turing machine with 3 tapes, each with 5 heads
[ ] Yes	[ ] No	A nondeterministic Turing machine with one head
[ ] Yes	[ ] No	A nondeterministic finite-state automaton (NFA)

(c) Which of the following languages are decidable?

Yes	No	$\varnothing$
Yes	No	$\{0^n1^n0^n1^n \mid n \geq 0\}$
Yes	No	$\{\langle M \rangle \mid M \text{ is a Turing machine}\}$
Yes	No	$\{\langle M \rangle \mid M \text{ accepts } \langle M \rangle \bullet \langle M \rangle\}$
Yes	No	$\{\langle M \rangle \mid M \text{ accepts a finite number of non-palindromes}\}$
Yes	No	$\{\langle M \rangle \mid M \text{ accepts } \varnothing\}$
Yes	No	$\{\langle M, w \rangle \mid M \text{ accepts } w^R\}$
Yes	No	$\{\langle M, w \rangle \mid M \text{ accepts } w \text{ after at most } \lvert w \rvert^2 \text{ transitions}\}$
Yes	No	$\{\langle M, w \rangle \mid M \text{ changes a blank on the tape to a non-blank, given input } w\}$
Yes	No	$\{\langle M, w \rangle \mid M \text{ changes a non-blank on the tape to a blank, given input } w\}$

(d) Let M be a standard Turing machine (with a single one-track tape and a single head) that ***decides*** the regular language 0^*1^*. Which of the following ***must*** be true?

Yes	No	Given an empty initial tape, M eventually halts.
Yes	No	M accepts the string 1111.
Yes	No	M rejects the string 0110.
Yes	No	M moves its head to the right at least once, given input 1100.
Yes	No	M moves its head to the right at least once, given input 0101.
Yes	No	M never accepts before reading a blank.
Yes	No	For some input string, M moves its head to the left at least once.
Yes	No	For some input string, M changes at least one symbol on the tape.
Yes	No	M always halts.
Yes	No	If M accepts a string w, it does so after at most $O(\lvert w \rvert^2)$ steps.

(e) Consider the following pair of languages:

- HAMILTONIANPATH := $\{G \mid G \text{ contains a Hamiltonian path}\}$
- CONNECTED := $\{G \mid G \text{ is connected}\}$

Which of the following ***must*** be true, assuming P≠NP?

Yes	No	CONNECTED ∈ NP
Yes	No	HAMILTONIANPATH ∈ NP
Yes	No	HAMILTONIANPATH is decidable.
Yes	No	There is no polynomial-time reduction from HAMILTONIANPATH to CONNECTED.
Yes	No	There is no polynomial-time reduction from CONNECTED to HAMILTONIANPATH.

(f) Suppose we want to prove that the following language is undecidable.

$$\text{ALWAYSHALTS} := \{\langle M \rangle \mid M \text{ halts on every input string}\}$$

Rocket J. Squirrel suggests a reduction from the standard halting language

$$\text{HALT} := \{\langle M, w \rangle \mid M \text{ halts on inputs } w\}.$$

Specifically, suppose there is a Turing machine AH that decides ALWAYSHALTS. Rocky claims that the following Turing machine H decides HALT. Given an arbitrary encoding $\langle M, w \rangle$ as input, machine H writes the encoding $\langle M' \rangle$ of a new Turing machine M' to the tape and passes it to AH, where M' implements the following algorithm:

```
M'(x):
  if M accepts w
    reject
  if M rejects w
    accept
```

Which of the following statements is true for all inputs $\langle M, w \rangle$?

Yes	No	If M accepts w, then M' halts on every input string.
Yes	No	If M diverges on w, then M' halts on every input string.
Yes	No	If M accepts w, then AH accepts $\langle M' \rangle$.
Yes	No	If M rejects w, then H rejects $\langle M, w \rangle$.
Yes	No	H decides the language HALT. (That is, Rocky's reduction is correct.)

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2. A ***relaxed 3-coloring*** of a graph G assigns each vertex of G one of three colors (for example, red, green, and blue), such that ***at most one*** edge in G has both endpoints the same color.

   (a) Give an example of a graph that has a relaxed 3-coloring, but does not have a proper 3-coloring (where every edge has endpoints of different colors).

   (b) ***Prove*** that it is NP-hard to determine whether a given graph has a relaxed 3-coloring.

3. Give a complete, formal, self-contained description of a DFA that accepts all strings in $\{0, 1\}^*$ containing at least ten 0s and at most ten 1s. Specifically:

   (a) What are the states of your DFA?

   (b) What is the start state of your DFA?

   (c) What are the accepting states of your DFA?

   (d) What is your DFA's transition function?

4. Suppose you are given three strings $A[1..n]$, $B[1..n]$, and $C[1..n]$. Describe and analyze an algorithm to find the maximum length of a common subsequence of all three strings. For example, given the input strings

$$A = \texttt{AxxBxxCDxEF}, \qquad B = \texttt{yyABCDyEyFy}, \qquad C = \texttt{zAzzBCDzEFz},$$

your algorithm should output the number 6, which is the length of the longest common subsequence `ABCDEF`.

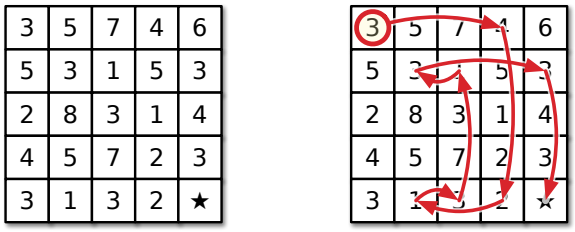
5. For each of the following languages over the alphabet $\Sigma = \{0, 1\}$, either ***prove*** that the language is regular, or ***prove*** that the language is not regular.

   (a) $\left\{ www \;\middle|\; w \in \Sigma^* \right\}$

   (b) $\left\{ wxw \;\middle|\; w, x \in \Sigma^* \right\}$

6. A ***number maze*** is an $n \times n$ grid of positive integers. A token starts in the upper left corner; your goal is to move the token to the lower-right corner. On each turn, you are allowed to move the token up, down, left, or right; the distance you may move the token is determined by the number on its current square. For example, if the token is on a square labeled 3, then you may move the token three steps up, three steps down, three steps left, or three steps right. However, you are never allowed to move the token off the edge of the board.

   Describe and analyze an efficient algorithm that either returns the minimum number of moves required to solve a given number maze, or correctly reports that the maze has no solution. For example, given the maze shown below, your algorithm would return the number 8.



A 5×5 number maze that can be solved in eight moves.

(scratch paper)

(scratch paper)

You may assume the following problems are NP-hard:

CIRCUITSAT: Given a boolean circuit, are there any input values that make the circuit output TRUE?

3SAT: Given a boolean formula in conjunctive normal form, with exactly three literals per clause, does the formula have a satisfying assignment?

MAXINDEPENDENTSET: Given an undirected graph G, what is the size of the largest subset of vertices in G that have no edges among them?

MAXCLIQUE: Given an undirected graph G, what is the size of the largest complete subgraph of G?

MINVERTEXCOVER: Given an undirected graph G, what is the size of the smallest subset of vertices that touch every edge in G?

3COLOR: Given an undirected graph G, can its vertices be colored with three colors, so that every edge touches vertices with two different colors?

HAMILTONIANPATH: Given an undirected graph G, is there a path in G that visits every vertex exactly once?

HAMILTONIANCYCLE: Given an undirected graph G, is there a cycle in G that visits every vertex exactly once?

DIRECTEDHAMILTONIANCYCLE: Given an directed graph G, is there a directed cycle in G that visits every vertex exactly once?

TRAVELINGSALESMAN: Given a graph G (either directed or undirected) with weighted edges, what is the minimum total weight of any Hamiltonian path/cycle in G?

DRAUGHTS: Given an $n \times n$ international draughts configuration, what is the largest number of pieces that can (and therefore must) be captured in a single move?

SUPER MARIO: Given an $n \times n$ level for Super Mario Brothers, can Mario reach the castle?

You may assume the following languages are undecidable:

$$\begin{aligned}
\textsc{SelfReject} &:= \{\langle M\rangle \mid M \text{ rejects } \langle M\rangle\} \\
\textsc{SelfAccept} &:= \{\langle M\rangle \mid M \text{ accepts } \langle M\rangle\} \\
\textsc{SelfHalt} &:= \{\langle M\rangle \mid M \text{ halts on } \langle M\rangle\} \\
\textsc{SelfDiverge} &:= \{\langle M\rangle \mid M \text{ does not halt on } \langle M\rangle\} \\
\textsc{Reject} &:= \{\langle M, w\rangle \mid M \text{ rejects } w\} \\
\textsc{Accept} &:= \{\langle M, w\rangle \mid M \text{ accepts } w\} \\
\textsc{Halt} &:= \{\langle M, w\rangle \mid M \text{ halts on } w\} \\
\textsc{Diverge} &:= \{\langle M, w\rangle \mid M \text{ does not halt on } w\} \\
\textsc{NeverReject} &:= \{\langle M\rangle \mid \textsc{Reject}(M) = \emptyset\} \\
\textsc{NeverAccept} &:= \{\langle M\rangle \mid \textsc{Accept}(M) = \emptyset\} \\
\textsc{NeverHalt} &:= \{\langle M\rangle \mid \textsc{Halt}(M) = \emptyset\} \\
\textsc{NeverDiverge} &:= \{\langle M\rangle \mid \textsc{Diverge}(M) = \emptyset\}
\end{aligned}$$

"CS 374": Algorithms and Models of Computation, Fall 2014

Final Exam (Version B) — December 16, 2014

Name:	
NetID:	
Section:	1 2 3

#	1	2	3	4	5	6	Total
Score							
Max	20	10	10	10	10	10	70
Grader							

- ***Don't panic!***
- Please print your name and your NetID and circle your discussion section in the boxes above.
- This is a closed-book, closed-notes, closed-electronics exam. If you brought anything except your writing implements and your two double-sided 8½" × 11" cheat sheets, please put it away for the duration of the exam. In particular, you may not use *any* electronic devices.
- **Please read the entire exam before writing anything.** Please ask for clarification if any question is unclear.
- **You have 180 minutes.**
- If you run out of space for an answer, continue on the back of the page, or on the blank pages at the end of this booklet, **but please tell us where to look.** Alternatively, feel free to tear out the blank pages and use them as scratch paper.
- **Please return your cheat sheets and all scratch paper with your answer booklet.**
- If you use a greedy algorithm, you must prove that it is correct to receive credit. **Otherwise, proofs are required only if we specifically ask for them.**
- As usual, answering any (sub)problem with "I don't know" (and nothing else) is worth 25% partial credit. **Yes, even for problem 1.** Correct, complete, but suboptimal solutions are *always* worth more than 25%. A blank answer is not the same as "I don't know".
- ***Good luck!*** And have a great winter break!

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1. For each of the following questions, indicate ***every*** correct answer by marking the "Yes" box, and indicate ***every*** incorrect answer by marking the "No" box. **Assume P $\neq$ NP.** If there is any other ambiguity or uncertainty, mark the "No" box. For example:

[X] Yes	[ ] No	$2+2=4$
[ ] Yes	[X] No	$x+y=5$
[ ] Yes	[X] No	3SAT can be solved in polynomial time.
[X] Yes	[ ] No	Jeff is not the Queen of England.

There are 40 yes/no choices altogether, each worth ½ point.

(a) Which of the following statements is true for ***every*** language $L \subseteq \{0,1\}^*$?

[ ] Yes	[ ] No	L is non-empty.
[ ] Yes	[ ] No	L is decidable or L is infinite (or both).
[ ] Yes	[ ] No	L is accepted by some DFA with 42 states if and only if L is accepted by some NFA with 42 states.
[ ] Yes	[ ] No	If L is regular, then $L \in$ NP.
[ ] Yes	[ ] No	L is decidable if and only if its complement $\overline{L}$ is undecidable.

(b) Which of the following computational models can simulate a deterministic Turing machine with three read/write heads, with at most polynomial slow-down in time, assuming P $\neq$ NP?

[ ] Yes	[ ] No	A C++ program
[ ] Yes	[ ] No	A deterministic Turing machine with one head
[ ] Yes	[ ] No	A deterministic Turing machine with 3 tapes, each with 5 heads
[ ] Yes	[ ] No	A nondeterministic Turing machine with one head
[ ] Yes	[ ] No	A nondeterministic finite-state automaton (NFA)

(c) Which of the following languages are decidable?

Yes	No	$\varnothing$
Yes	No	$\{ww \mid w \text{ is a palindrome}\}$
Yes	No	$\{\langle M \rangle \mid M \text{ is a Turing machine}\}$
Yes	No	$\{\langle M \rangle \mid M \text{ accepts } \langle M \rangle \bullet \langle M \rangle\}$
Yes	No	$\{\langle M \rangle \mid M \text{ accepts an infinite number of palindromes}\}$
Yes	No	$\{\langle M \rangle \mid M \text{ accepts } \varnothing\}$
Yes	No	$\{\langle M, w \rangle \mid M \text{ accepts } www\}$
Yes	No	$\{\langle M, w \rangle \mid M \text{ accepts } w \text{ after } \textbf{at least } \lvert w \rvert^2 \text{ transitions}\}$
Yes	No	$\{\langle M, w \rangle \mid M \text{ changes a non-blank on the tape to a blank, given input } w\}$
Yes	No	$\{\langle M, w \rangle \mid M \text{ changes a blank on the tape to a non-blank, given input } w\}$

(d) Let M be a standard Turing machine (with a single one-track tape and a single head) such that $\textsc{Accept}(M)$ is the regular language 0^*1^*. Which of the following ***must*** be true?

Yes	No	Given an empty initial tape, M eventually halts.
Yes	No	M accepts the string 1111.
Yes	No	M rejects the string 0110.
Yes	No	M moves its head to the right at least once, given input 1100.
Yes	No	M moves its head to the right at least once, given input 0101.
Yes	No	M must read a blank before it accepts.
Yes	No	For some input string, M moves its head to the left at least once.
Yes	No	For some input string, M changes at least one symbol on the tape.
Yes	No	M always halts.
Yes	No	If M accepts a string w, it does so after at most $O(\lvert w \rvert^2)$ steps.

(e) Consider the following pair of languages:

- HAMILTONIANPATH := $\{G \mid G \text{ contains a Hamiltonian path}\}$
- CONNECTED := $\{G \mid G \text{ is connected}\}$

Which of the following ***must*** be true, assuming P≠NP?

Yes	No	CONNECTED ∈ NP
Yes	No	HAMILTONIANPATH ∈ NP
Yes	No	HAMILTONIANPATH is undecidable.
Yes	No	There is a polynomial-time reduction from HAMILTONIANPATH to CONNECTED.
Yes	No	There is a polynomial-time reduction from CONNECTED to HAMILTONIANPATH.

(f) Suppose we want to prove that the following language is undecidable.

$$\text{ALWAYSHALTS} := \{\langle M \rangle \mid M \text{ halts on every input string}\}$$

Bullwinkle J. Moose suggests a reduction from the standard halting language

$$\text{HALT} := \{\langle M, w \rangle \mid M \text{ halts on inputs } w\}.$$

Specifically, suppose there is a Turing machine AH that decides ALWAYSHALTS. Bullwinkle claims that the following Turing machine H decides HALT. Given an arbitrary encoding $\langle M, w \rangle$ as input, machine H writes the encoding $\langle M' \rangle$ of a new Turing machine M' to the tape and passes it to AH, where M' implements the following algorithm:

```
M'(x):
  if M accepts w
    reject
  if M rejects w
    accept
```

Which of the following statements is true for all inputs $\langle M, w \rangle$?

Yes	No	If M accepts w, then M' halts on every input string.
Yes	No	If M rejects w, then M' halts on every input string.
Yes	No	If M rejects w, then H rejects $\langle M, w \rangle$.
Yes	No	If M diverges on w, then H diverges on $\langle M, w \rangle$.
Yes	No	H does not correctly decide the language HALT. (That is, Bullwinkle's reduction is incorrect.)

---

2. A ***near-Hamiltonian cycle*** in a graph G is a closed walk in G that visits one vertex exactly twice and every other vertex exactly once.

   (a) Give an example of a graph that contains a near-Hamiltonian cycle, but does not contain a Hamiltonian cycle (which visits every vertex exactly once).

   (b) ***Prove*** that it is NP-hard to determine whether a given graph contains a near-Hamiltonian cycle.

3. Give a complete, formal, self-contained description of a DFA that accepts all strings in $\{0,1\}^*$ such that every fifth bit is `0` and the length is *not* divisible by 12. For example, your DFA should accept the strings `11110111101` and `11`. Specifically:

   (a) What are the states of your DFA?

   (b) What is the start state of your DFA?

   (c) What are the accepting states of your DFA?

   (d) What is your DFA's transition function?

4. Suppose you are given three strings $A[1..n]$, $B[1..n]$, and $C[1..n]$. Describe and analyze an algorithm to find the maximum length of a common subsequence of all three strings. For example, given the input strings

$$A = \texttt{AxxBxxCDxEF}, \qquad B = \texttt{yyABCDyEyFy}, \qquad C = \texttt{zAzzBCDzEFz},$$

your algorithm should output the number 6, which is the length of the longest common subsequence `ABCDEF`.

5. For each of the following languages over the alphabet $\Sigma = \{0, 1\}$, either ***prove*** that the language is regular, or ***prove*** that the language is not regular.

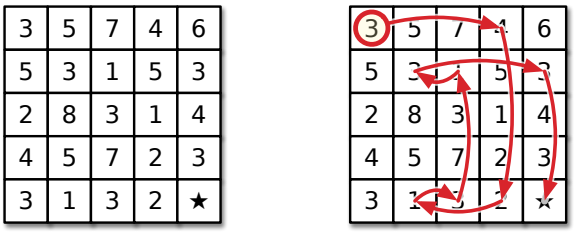
   (a) $\left\{ www \;\middle|\; w \in \Sigma^* \right\}$

   (b) $\left\{ wxw \;\middle|\; w, x \in \Sigma^* \right\}$

6. A ***number maze*** is an $n \times n$ grid of positive integers. A token starts in the upper left corner; your goal is to move the token to the lower-right corner.

   - On each turn, you are allowed to move the token up, down, left, or right.
   - The distance you may move the token is determined by the number on its current square. For example, if the token is on a square labeled 3, then you may move the token three steps up, three steps down, three steps left, or three steps right.
   - However, you are never allowed to move the token off the edge of the board. In particular, if the current number is too large, you may not be able to move at all.

   Describe and analyze an efficient algorithm that either returns the minimum number of moves required to solve a given number maze, or correctly reports that the maze has no solution. For example, given the maze shown below, your algorithm would return the number 8.



A 5 × 5 number maze that can be solved in eight moves.

(scratch paper)

(scratch paper)

You may assume the following problems are NP-hard:

CircuitSat: Given a boolean circuit, are there any input values that make the circuit output True?

3Sat: Given a boolean formula in conjunctive normal form, with exactly three literals per clause, does the formula have a satisfying assignment?

MaxIndependentSet: Given an undirected graph G, what is the size of the largest subset of vertices in G that have no edges among them?

MaxClique: Given an undirected graph G, what is the size of the largest complete subgraph of G?

MinVertexCover: Given an undirected graph G, what is the size of the smallest subset of vertices that touch every edge in G?

3Color: Given an undirected graph G, can its vertices be colored with three colors, so that every edge touches vertices with two different colors?

HamiltonianPath: Given an undirected graph G, is there a path in G that visits every vertex exactly once?

HamiltonianCycle: Given an undirected graph G, is there a cycle in G that visits every vertex exactly once?

DirectedHamiltonianCycle: Given an directed graph G, is there a directed cycle in G that visits every vertex exactly once?

TravelingSalesman: Given a graph G (either directed or undirected) with weighted edges, what is the minimum total weight of any Hamiltonian path/cycle in G?

Draughts: Given an $n \times n$ international draughts configuration, what is the largest number of pieces that can (and therefore must) be captured in a single move?

Super Mario: Given an $n \times n$ level for Super Mario Brothers, can Mario reach the castle?

You may assume the following languages are undecidable:

$$\textsc{SelfReject} := \{\langle M \rangle \mid M \text{ rejects } \langle M \rangle\}$$
$$\textsc{SelfAccept} := \{\langle M \rangle \mid M \text{ accepts } \langle M \rangle\}$$
$$\textsc{SelfHalt} := \{\langle M \rangle \mid M \text{ halts on } \langle M \rangle\}$$
$$\textsc{SelfDiverge} := \{\langle M \rangle \mid M \text{ does not halt on } \langle M \rangle\}$$
$$\textsc{Reject} := \{\langle M, w \rangle \mid M \text{ rejects } w\}$$
$$\textsc{Accept} := \{\langle M, w \rangle \mid M \text{ accepts } w\}$$
$$\textsc{Halt} := \{\langle M, w \rangle \mid M \text{ halts on } w\}$$
$$\textsc{Diverge} := \{\langle M, w \rangle \mid M \text{ does not halt on } w\}$$
$$\textsc{NeverReject} := \{\langle M \rangle \mid \textsc{Reject}(M) = \emptyset\}$$
$$\textsc{NeverAccept} := \{\langle M \rangle \mid \textsc{Accept}(M) = \emptyset\}$$
$$\textsc{NeverHalt} := \{\langle M \rangle \mid \textsc{Halt}(M) = \emptyset\}$$
$$\textsc{NeverDiverge} := \{\langle M \rangle \mid \textsc{Diverge}(M) = \emptyset\}$$