This exam lasts 180 minutes.
Write your answers in the separate answer booklet.
Please return this question handout and your cheat sheets with your answers.

1. Suppose you are given a sorted array of n distinct numbers that has been *rotated* k steps, for some ***unknown*** integer k between 1 and $n-1$. That is, you are given an array $A[1..n]$ such that the prefix $A[1..k]$ is sorted in increasing order, the suffix $A[k+1..n]$ is sorted in increasing order, and $A[n] < A[1]$. Describe and analyze an algorithm to compute the unknown integer k.

   For example, given the following array as input, your algorithm should output the integer 10.

9	13	16	18	19	23	28	31	37	42	−4	0	2	5	7	8

2. You are hired as a cyclist for the Giggle Highway View project, which will provide street-level images along the entire US national highway system. As a pilot project, you are asked to ride the Giggle Highway-View Fixed-Gear Carbon-Fiber Bicycle from "the Giggleplex" in Portland, Oregon to "Giggleburg" in Williamsburg, Brooklyn, New York.

   You are a hopeless caffeine addict, but like most Giggle employees you are also a coffee snob; you only drink independently roasted organic shade-grown single-origin espresso. After each espresso shot, you can bike up to L miles before suffering a caffeine-withdrawal migraine.

   Giggle helpfully provides you with a map of the United States, in the form of an undirected graph G, whose vertices represent coffee shops that sell independently roasted organic shade-grown single-origin espresso, and whose edges represent highway connections between them. Each edge e is labeled with the length $\ell(e)$ of the corresponding stretch of highway. Naturally, there are espresso stands at both Giggle offices, represented by two specific vertices s and t in the graph G.

   (a) Describe and analyze an algorithm to determine whether it is possible to bike from the Giggleplex to Giggleburg without suffering a caffeine-withdrawal migraine.

   (b) When you report to your supervisor (whom Giggle recently hired away from competitor Yippee!) that the ride is impossible, she demands to look at your map. "Oh, I see the problem; there are no *Starbucks* on this map!" As you look on in horror, she hands you an updated graph G' that includes a vertex for every Starbucks location in the United States, helpfully marked in Starbucks Green (Pantone® 3425 C).

   Describe and analyze an algorithm to find the minimum number of Starbucks locations you must visit to bike from the Giggleplex to Giggleburg without suffering a caffeine-withdrawal migraine. More formally, your algorithm should find the minimum number of green vertices on any path in G' from s to t that uses only edges of length at most L.

3. Suppose you are given a collection of up-trees representing a partition of the set $\{1, 2, \ldots, n\}$ into subsets. **You have no idea how these trees were constructed.** You are also given an array $node[1..n]$, where $node[i]$ is a pointer to the up-tree node containing element i. Your task is to create a new array $label[1..n]$ using the following algorithm:

```
LABELEVERYTHING:
  for i ← 1 to n
      label[i] ← FIND(node[i])
```

Recall that there are two natural ways to implement FIND: simple pointer-chasing and pointer-chasing with path compression. Pseudocode for both methods is shown below.

```
FIND(x):
  while x ≠ parent(x)
      x ← parent(x)
  return x
```

Without path compression

```
FIND(x):
  if x ≠ parent(x)
      parent(x) ← FIND(parent(x))
  return parent(x)
```

With path compression

(a) What is the worst-case running time of LABELEVERYTHING if we implement FIND *without* path compression?

(b) ***Prove*** that if we implement FIND using path compression, LABELEVERYTHING runs in $O(n)$ time in the worst case.

4. Congratulations! You have successfully conquered Camelot, transforming the former battle-scarred kingdom with an anarcho-syndicalist commune, where citizens take turns to act as a sort of executive-officer-for-the-week, but with all the decisions of that officer ratified at a special bi-weekly meeting, by a simple majority in the case of purely internal affairs, but by a two-thirds majority, in the case of more major....

As a final symbolic act, you order the Round Table (surprisingly, an actual circular table) to be split into pizza-like wedges and distributed to the citizens of Camelot as trophies. Each citizen has submitted a request for an angular wedge of the table, specified by two angles—for example, Sir Robin the Brave might request the wedge from 17° to 42°. Each citizen will be happy if and only if they receive *precisely* the wedge that they requested. Unfortunately, some of these ranges overlap, so satisfying *all* the citizens' requests is simply impossible. Welcome to politics.

Describe and analyze an algorithm to find the maximum number of requests that can be satisfied.

5. The NSA has established several monitoring stations around the country, each one conveniently hidden in the back of a Starbucks. Each station can monitor up to 42 cell-phone towers, but can only monitor cell-phone towers within a 20-mile radius. To ensure that every cell-phone call is recorded even if some stations malfunction, the NSA requires each cell-phone tower to be monitored by at least 3 different stations.

   Suppose you know that there are n cell-phone towers and m monitoring stations, and you are given a function DISTANCE(i, j) that returns the distance between the ith tower and the jth station in $O(1)$ time. Describe and analyze an algorithm that either computes a valid assignment of cell-phone towers to monitoring stations, or reports correctly that there is no such assignment (in which case the NSA will build another Starbucks).

6. Consider the following closely related problems:

   - HAMILTONIANPATH: Given an undirected graph G, determine whether G contains a ***path*** that visits every vertex of G exactly once.
   - HAMILTONIANCYCLE: Given an undirected graph G, determine whether G contains a ***cycle*** that visits every vertex of G exactly once.

   Describe a polynomial-time reduction from HAMILTONIANCYCLE to HAMILTONIANPATH. ***Prove*** your reduction is correct. *[Hint: A polynomial-time reduction is allowed to call the black-box subroutine more than once.]*

7. An array $X[1..n]$ of distinct integers is ***wobbly*** if it alternates between increasing and decreasing: $X[i] < X[i+1]$ for every odd index i, and $X[i] > X[i+1]$ for every even index i. For example, the following 16-element array is wobbly:

12	13	0	16	13	31	5	7	−1	23	8	10	−4	37	17	42

   Describe and analyze an algorithm that permutes the elements of a given array to make the array wobbly.

You may use the following algorithms as black boxes:

RANDOM(k): Given any positive integer k, return an integer chosen independently and uniformly at random from the set $\{1, 2, \ldots, k\}$ in $O(1)$ time.

ORLINMAXFLOW(V, E, c, s, t): Given a directed graph $G = (V, E)$, a capacity function $c : E \to \mathbb{R}^+$, and vertices s and t, return a maximum (s, t)-flow in G in $O(VE)$ time. If the capacities are integral, so is the returned maximum flow.

Any other algorithm that we described in class.

You may assume the following problems are NP-hard:

CIRCUITSAT: Given a boolean circuit, are there any input values that make the circuit output TRUE?

PLANARCIRCUITSAT: Given a boolean circuit drawn in the plane so that no two wires cross, are there any input values that make the circuit output TRUE?

3SAT: Given a boolean formula in conjunctive normal form, with exactly three literals per clause, does the formula have a satisfying assignment?

MAXINDEPENDENTSET: Given an undirected graph G, what is the size of the largest subset of vertices in G that have no edges among them?

MAXCLIQUE: Given an undirected graph G, what is the size of the largest complete subgraph of G?

MINVERTEXCOVER: Given an undirected graph G, what is the size of the smallest subset of vertices that touch every edge in G?

MINSETCOVER: Given a collection of subsets $S_1, S_2, \ldots, S_m$ of a set S, what is the size of the smallest subcollection whose union is S?

MINHITTINGSET: Given a collection of subsets $S_1, S_2, \ldots, S_m$ of a set S, what is the size of the smallest subset of S that intersects every subset S_i?

3COLOR: Given an undirected graph G, can its vertices be colored with three colors, so that every edge touches vertices with two different colors?

HAMILTONIANCYCLE: Given a graph G, is there a cycle in G that visits every vertex exactly once?

HAMILTONIANPATH: Given a graph G, is there a path in G that visits every vertex exactly once?

TRAVELINGSALESMAN: Given a graph G with weighted edges, what is the minimum total weight of any Hamiltonian path/cycle in G?

STEINERTREE: Given an undirected graph G with some of the vertices marked, what is the minimum number of edges in a subtree of G that contains every marked vertex?

SUBSETSUM: Given a set X of positive integers and an integer k, does X have a subset whose elements sum to k?

PARTITION: Given a set X of positive integers, can X be partitioned into two subsets with the same sum?

3PARTITION: Given a set X of $3n$ positive integers, can X be partitioned into n three-element subsets, all with the same sum?

DRAUGHTS: Given an $n \times n$ international draughts configuration, what is the largest number of pieces that can (and therefore must) be captured in a single move?

DOGE: Such N. Many P. Wow.