1. **Updating a maximum flow**

   Suppose you are given a directed graph $G = (V, E)$, with a positive integer capacity c_e on each edge e, a designated source $s \in V$, and a designated sink $t \in V$. You are also given a maximum $s - t$ flow in G, defined by a flow value f_e on each edge e. The flow $\{f_e\}$ is *acyclic*: There is no cycle in G on which all edges carry positive flow.

   Now suppose we pick a specific edge $e^* \in E$ and reduce its capacity by 1 unit. Show how to find a maximum flow in the resulting capacitated graph in time $O(m + n)$, where m is the number of edges in G and n is the number of nodes.

2. **Cooking Schedule**

   You live in a cooperative apartment with n other people. The co-op needs to schedule cooks for the next n days, so that each person cooks one day and each day there is one cook. In addition, each member of the co-op has a list of days they are available to cook (and is unavailable to cook on the other days).

   Because of your superior CS473 skills, the co-op selects you to come up with a schedule for cooking, so that everyone cooks on a day they are available.

   (a) Describe a bipartite graph G so that G has a perfect matching if and only if there is a feasible schedule for the co-op.

   (b) A friend of yours tried to help you out by coming up with a cooking schedule. Unfortunately, when you look at the schedule he created, you notice a big problem. $n - 2$ of the people are scheduled for different nights on which they are available: no problem there. But the remaining two people are assigned to cook on the same night (and no one is assigned to the last night).

   You want to fix your friend's mistake, but without having to recompute everything from scratch. Show that it's possible, using his "almost correct" schedule to decide in $O(n^2)$ time whether there exists a feasible schedule.

3. **Disjoint paths in a digraph**

   Let $G = (V, E)$ be a directed graph, and suppose that for each node v, the number of edges into v is equal to the number of edges out of v. That is, for all v,

   $$|\{(u, v) : (u, v) \in E\}| = |\{(v, w) : (v, w) \in E\}|.$$

   Let x, y be two nodes of G, and suppose that there exist k mutually edge-disjoint paths from x to y. Under these conditions, does it follow that there exist k mutually edge-disjoint paths from y to x. Give a proof or a counterexample with explanation.