You have 90 minutes to answer four of these questions.

Write your answers in the separate answer booklet.

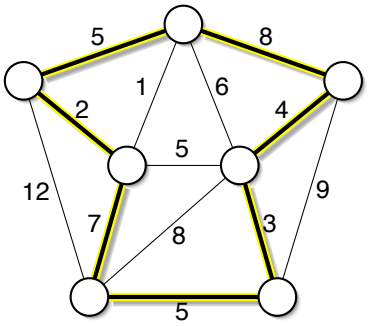
You may take the question sheet with you when you leave.

Chernoff Bounds: If X is the sum of independent indicator variables and $\mu = \mathrm{E}[X]$, then the following inequalities hold for any $\delta > 0$:

$$\Pr[X < (1-\delta)\mu] < \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu} \qquad \Pr[X > (1+\delta)\mu] < \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}$$

1. Describe and analyze an algorithm that randomly shuffles an array $X[1..n]$, so that each of the $n!$ possible permutations is equally likely, in $O(n)$ time. (Assume that the subroutine RANDOM(m) returns an integer chosen uniformly at random from the set $\{1, 2, , \ldots, m\}$ in $O(1)$ time.)

2. Let G be an undirected graph with weighted edges. A *heavy Hamiltonian cycle* is a cycle C that passes through each vertex of G exactly once, such that the total weight of the edges in C is at least half of the total weight of all edges in G. Prove that deciding whether a graph has a heavy Hamiltonian cycle is NP-complete.



A heavy Hamiltonian cycle. The cycle has total weight 34; the graph has total weight 67.

3. A sequence of numbers $\langle a_1, a_2, a_3, \ldots a_n \rangle$ is *oscillating* if $a_i < a_{i+1}$ for every *odd* index i and $a_i > a_{i+1}$ for every *even* index i. Describe and analyze an efficient algorithm to compute the longest oscillating subsequence in a sequence of n integers.

4. This problem asks you to how to efficiently modify a maximum flow if one of the edge capacities changes. Specifically, you are given a directed graph $G = (V, E)$ with capacities $c : E \to \mathbb{Z}_+$, and a maximum flow $F : E \to \mathbb{Z}$ from some vertex s to some other vertex t in G. Describe and analyze efficient algorithms for the following operations:

   (a) INCREMENT(e) — Increase the capacity of edge e by 1 and update the maximum flow F.

   (b) DECREMENT(e) — Decrease the capacity of edge e by 1 and update the maximum flow F.

   Both of your algorithms should be significantly faster than recomputing the maximum flow from scratch.

5.

6. Let $G = (V, E)$ be an undirected graph, each of whose vertices is colored either red, green, or blue. An edge in G is *boring* if its endpoints have the same color, and *interesting* if its endpoints have different colors. The *most interesting 3-coloring* is the 3-coloring with the maximum number of interesting edges, or equivalently, with the fewest boring edges.

   (a) Prove that it is NP-hard to compute the most interesting 3-coloring of a graph. *[Hint: There is a one-line proof. Use one of the NP-hard problems described in class.]*

   (b) Let $zzz(G)$ denote the number of boring edges in the most interesting 3-coloring of a graph G. Prove that it is NP-hard to approximate $zzz(G)$ within a factor of $10^{10^{100}}$. *[Hint: There is a one-line proof.]*

   (c) Let $wow(G)$ denote the number of interesting edges in the most interesting 3-coloring of G. Suppose we assign each vertex in G a *random* color from the set $\{\text{red}, \text{green}, \text{blue}\}$. Prove that the expected number of interesting edges is at least $\frac{2}{3}wow(G)$.

7.